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Gaussian Generalized Adrien Numbers

Feyza Demirci [a*] and Yüksel Soykan [a]

[a] *Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this study, we introduce the concept of Gaussian Generalized Adrien numbers, a novel extension within the framework of special number sequences. Our focus centers on two particular instances: the Gaussian Adrien numbers and the Gaussian Adrien-Lucas numbers. We systematically investigate and establish fundamental properties of these sequences, including closed-form identities, recurrence relations, matrix formulations, and Binet-type expressions. Additionally, we derive their generating functions, explore their connections with exponential functions, and present analogues of Simson's and summation formulas. These results contribute to a deeper algebraic and combinatorial understanding of the Gaussian extensions of Adrien-type numbers and open pathways for further research in number theory and related fields.

Keywords: Adrien numbers; Adrien-Lucas numbers; gaussian Adrien numbers; gaussian Adrien-Lucas numbers.

**Corresponding author: E-mail: demircfeyza6776@gmail.com;*

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1 INTRODUCTION: AN OVERVIEW OF GENERAL LINEAR RECURRENCE RELATIONS

Second-order, third-order, and fourth-order linear recurrence relations are particular cases of the more general k-th order linear recurrence relations. These describe sequences in which each term is a linear combination of a fixed number of preceding terms. A homogeneous linear recurrence relation of order k is defined as:

$$a_n = A_1a_{n-1} + A_2a_{n-2} + \cdots + A_ka_{n-k}, \quad \text{for } n \geq k,$$

where $A_1, A_2, \ldots, A_k$ are constant coefficients and the initial terms $a_0, a_1, \ldots, a_{k-1}$ are given.

Examples of Specific Orders:

- **Second-order:**

  $$a_n = Aa_{n-1} + Ba_{n-2}$$

  Characteristic equation:

  $$x^2 - Ax - B = 0$$

- **Third-order:**

  $$a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$$

  Characteristic equation:

  $$x^3 - Ax^2 - Bx - C = 0$$

  Fourth-order:

  $$a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3} + Da_{n-4}$$

  Characteristic equation:

  $$x^4 - Ax^3 - Bx^2 - Cx - D = 0$$

  The general solution to a linear recurrence relation depends on the roots of its characteristic equation. If the characteristic polynomial has k distinct roots $r_1, r_2, \ldots, r_k$, then the solution can be expressed as:

  $$a_n = \alpha_1r_1^n + \alpha_2r_2^n + \cdots + \alpha_kr_k^n,$$

  where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants determined by the initial conditions. If some roots are repeated or complex, the solution will include polynomial or trigonometric modifications such as:

  $$a_n = (P(n)r^n) \quad or \quad a_n = r^n\left(\alpha\cos(n\theta) + \beta\sin(n\theta)\right),$$

  depending on whether the roots are repeated real or complex conjugates.

An inhomogeneous linear recurrence includes a non-zero function on the right-hand side:

$$a_n = A_1a_{n-1} + \cdots + A_ka_{n-k} + f(n),$$

where $f(n)$ is a function of n. The general solution is the sum of the homogeneous solution and a particular solution of the nonhomogeneous relation.

Higher-order recurrence relations are fundamental in:

- **Combinatorics:** Enumeration of partitions, tilings, and paths.
- **Computer Science:** Algorithm analysis and dynamic programming.
- **Mathematical Physics:** Discrete dynamical systems and numerical schemes.
- **Number Theory:** Generalizations of Fibonacci, Lucas, and other integer sequences.

2 BACKGROUND ON ADRIEN NUMBERS

In this section, we present key foundational results on Adrien numbers, which are governed by a fourth-order homogeneous recurrence relation.

The generalized Adrien sequence $\{W_n\}_{n\geq0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq0}$ is defined by the fourth-order recurrence relation as

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-4}, \qquad (2.1)$$

with the initial values W_0, W_1, W_2, W_3 not all being zero.

The sequence $\{W_n\}_{n\geq0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-2)} + 3W_{-(n-3)} - W_{-(n-4)},$$

for $n = 1, 2, 3, ...$ Hence, recurrence (2.1) holds for all integer n.Soykan has conducted a study on this particular sequence, for more details, see (Soykan, 2023a).

Characteristic equation of $\{W_n\}$ is

$$z^4 - 3z^3 + z^2 + 1 = (z^3 - 2z^2 - z - 1)(z - 1) = 0,$$

whose roots are

$$\begin{aligned}
\alpha &= \frac{2}{3} + \left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\beta &= \frac{2}{3} + \omega\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega^2\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\gamma &= \frac{2}{3} + \omega^2\left(\frac{61}{54} + \sqrt{\frac{29}{36}}\right)^{1/3} + \omega\left(\frac{61}{54} - \sqrt{\frac{29}{36}}\right)^{1/3}, \\
\delta &= 1,
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 3, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= 1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 0, \\
\alpha\beta\gamma\delta &= 1.
\end{aligned}$$

Note also that

$$\begin{aligned}
\alpha + \beta + \gamma &= 2, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

$$\begin{aligned}
p_1 &= W_3 - (\beta + \gamma + \delta)W_2 + (\beta\gamma + \beta\delta + \gamma\delta)W_1 - \beta\gamma\delta W_0, \\
p_2 &= W_3 - (\alpha + \gamma + \delta)W_2 + (\alpha\gamma + \alpha\delta + \gamma\delta)W_1 - \alpha\gamma\delta W_0, \\
p_3 &= W_3 - (\alpha + \beta + \delta)W_2 + (\alpha\beta + \alpha\delta + \beta\delta)W_1 - \alpha\beta\delta W_0, \\
p_4 &= W_3 - (\alpha + \beta + \gamma)W_2 + (\alpha\beta + \alpha\gamma + \beta\gamma)W_1 - \alpha\beta\gamma W_0,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}, \\
A_2 &= \frac{p_2}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\
A_3 &= \frac{p_3}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\
A_4 &= \frac{p_4}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}.
\end{aligned}$$

For $n = 1, 2, 3....$ Hence, recurrence (2.1)is true for all integer n.

For the fourth-order reccurrance relations has been studied by many authors, for more detail see (Soykan, 2020b, 2023c,e,b, 2025, 2023d, 2021f,e).

We now present Binet's formula for the generalized Adrien numbers.

Theorem 2.1. *2023 Binet formula of generalized Adrien numbers can be presented as follows:*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)\alpha^n}{4\alpha^2 + 3\alpha - 1} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)\beta^n}{4\beta^2 + 3\beta - 1} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)\gamma^n}{4\gamma^2 + 3\gamma - 1} \\ &+ \frac{W_3 - 2W_2 - W_1 - W_0}{-3}. \end{aligned}$$

Now we define two special cases of the sequence $\{W_n\}$ as follows: The Adrien sequence $\{A_n\}_{n\geq 0}$ and the Adrien-Lucas sequence $\{B_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations as:

$$A_n = 3A_{n-1} - A_{n-2} - A_{n-4}\ ,\ A_0 = 0, A_1 = 1, A_2 = 3, A_3 = 8,\ n \geq 4, \quad (2.2)$$

$$B_n = 3B_{n-1} - B_{n-2} - B_{n-4}\ ,\ B_0 = 4, B_1 = 3, B_2 = 7, B_3 = 18,\ n \geq 4. \quad (2.3)$$

The sequences $\{A_n\}_{n\geq 0},\ \{B_n\}_{n\geq 0},$ can be extended to negative subscripts by defining,

$$\begin{aligned} A_{-n} &= -A_{-(n-2)} + 3A_{-(n-3)} - A_{-(n-4)}, \\ B_{-n} &= -B_{-(n-2)} + 3B_{-(n-3)} - B_{-(n-4)}. \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (2.2)-(2.3) hold for all integer n.Binet's formulas as follows.

Now we introduce Binet's formula of Adrien and Adrien-Lucas numbers.

Corollary 2.2. *For all integers n, Binet's formula of Adrien and Adrien-Lucas numbers are*

$$A_n = \frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3},$$

and

$$B_n = \alpha^n + \beta^n + \gamma^n + 1.$$

respectively.

Lemma 2.3. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Adrien sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - 3W_0)z + (W_2 - 3W_1 + W_0)z^2 + (W_3 - 3W_2 + W_1)z^3}{1 - 3z + z^2 + z^4}.$$

Proof. Take $r = 3, s = -1, t = 0, u = -1$ in Lemma (Soykan, 2023a). □

Next, we give some information about Gaussian sequences from literature.

- Horadam (1963) introduced Gaussian Fibonacci numbers and defined by

$$GF_n = F_n + iF_{n-1}$$

  where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (in fact, he defined these numbers as $GF_n = F_n + iF_{n+1}$ and he called them as complex Fibonacci numbers.

- Pethe and Horadam (1986) introduced Gaussian generalized Fibonacci numbers by

$$GF_n = F_n + iF_{n-1},$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Halıcı and Öz (2016) studied Gaussian Pell and Pell Lucas numbers by written , respectively,

$$\begin{aligned} GP_n &= P_n + iP_{n-1}, \\ GQ_n &= Q_n + iQ_{n-1}. \end{aligned}$$

  We give some Gaussian numbers with second third recurence relations.

- Yılmaz and Soykan (Yılmaz and Soykan, 2023) studied Gaussian Guglielmo and Guglielmo-Lucas numbers by written respectively,

$$\begin{aligned} GT_n &= T_n + iT_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \end{aligned}$$

where $T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, T_0 = 0, T_1 = 1, T_2 = 3$, and $H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3},$ $H_0 = 3, H_1 = 3, H_2 = 3.$

- Dikmen (2025) presented Gaussian Leonardo and Leonardo-Lucas numbers by written respectively,

$$\begin{aligned} Gl_n &= l_n + il_{n-1}, \\ GH_n &= H_n + iH_{n-1}, \end{aligned}$$

where $l_n = 2l_{n-1} - l_{n-3}, l_0 = 1, l_1 = 1, l_2 = 3$, and $H_n = 2H_{n-1} - H_{n-3}, H_0 = 3, H_1 = 2, H_2 = 4.$

- Ayrılma and Soykan (2025) presented Gaussian Edouard and Edouard-Lucas numbers by written respectively,

$$\begin{aligned} GE_n &= E_n + iE_{n-1}, \\ GK_n &= K_n + iK_{n-1}, \end{aligned}$$

where $E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, E_0 = 0, E_1 = 1, E_2 = 7$, and $K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3},$ $K_0 = 3, K_1 = 7, K_2 = 35.$

- Soykan et al. (2023) describe Gaussian Bigollo and Bigollo-Lucas numbers by written respectively,

$$\begin{aligned} GB_n &= B_n + iB_{n-1}, \\ GC_n &= C_n + iC_{n-1}, \end{aligned}$$

where $B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, B_0 = 0, B_1 = 1, B_2 = 4$ and $C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3},$ $C_0 = 3, C_1 = 4, C_2 = 6.$

- Eren and Soykan (2023) describe Gaussian Woodall and Woodall-Lucas numbers by written respectively,

$$\begin{aligned} GR_n &= R_n + iR_{n-1}, \\ GC_n &= C_n + iC_{n-1}, \end{aligned}$$

where $R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3},\ R_0 = -1, R_1 = 1, R_2 = 7,,$ and $C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3},$ $C_0 = 1, C_1 = 3, C_2 = 9.$

Next, we give the exponential generating function of $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ of the sequence W_n.

Lemma 2.4. *Suppose that* $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ *is the exponential generating function of the generalized Adrien sequence* $\{W_n\}$.

Then $\sum\limits_{n=0}^{\infty} W_n \frac{x^n}{n!}$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}. \end{aligned}$$

Proof: Using the Binet's formula of generating Adrien numbers we get

$$\begin{aligned} \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (\frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} + \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} + \frac{W_3 - 2W_2 - W_1 - W_0}{-3}) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{p_1\alpha^n}{4\alpha^2 + 3\alpha - 1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{p_2\beta^n}{4\beta^2 + 3\beta - 1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{p_3\gamma^n}{4\gamma^2 + 3\gamma - 1} \frac{x^n}{n!} \\ &+ \sum_{n=0}^{\infty} (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) \frac{x^n}{n!} \\ &= \frac{p_1}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} + \frac{p_2}{4\beta^2 + 3\beta - 1} e^{\beta x} + \frac{p_3}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} + \frac{W_3 - 2W_2 - W_1 - W_0}{-3} e^{\delta x}. \square \end{aligned}$$

The previous Lemma 2.4 gives the following results as particular examples.

Corollary 2.5. *Exponential generating function of Adrien and Adrien-Lucas numbers*

a) $\sum\limits_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} ((\frac{(2\alpha^2 + \alpha + 1)\alpha^n}{4\alpha^2 + 3\alpha - 1} + \frac{(2\beta^2 + \beta + 1)\beta^n}{4\beta^2 + 3\beta - 1} + \frac{(2\gamma^2 + \gamma + 1)\gamma^n}{4\gamma^2 + 3\gamma - 1} - \frac{1}{3}) \frac{x^n}{n!}$

$$=(\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}e^{\gamma x}-\frac{1}{3}e^x)$$

b) $\sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!} = \sum\limits_{n=0}^{\infty} (\alpha^n+\beta^n+\gamma^n+1+)\frac{x^n}{n!} = e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^x.$

3 GENERALIZED GAUSSIAN ADRIEN NUMBERS

In this section, we introduces Gaussian numbers and explores some of their key properties including Binet's formula and generating functions.

Gaussian generalized Adrien numbers $\{GW_n\}_{n\geq 0} = \{GW_n(GW_0, GW_1, GW_2, GW_3)\}_{n\geq 0}$ are defined by

$$GW_n = 3GW_{n-1} - GW_{n-2} - GW_{n-4}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} GW_0 &= W_0 + i(3W_2 - W_1 - W_3), \\ GW_1 &= W_1 + iW_0, \\ GW_2 &= W_2 + iW_1, \\ GW_3 &= W_3 + iW_2. \end{aligned}$$

not all being zero. The sequences $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -GW_{-(n-2)} + 3GW_{-(n-3)} - GW_{-(n-4)}. \tag{3.2}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (3.1) hold for all integer n. Note that for all integers n, we get

$$GW_n = W_n + iW_{n-1}, \tag{3.3}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}. \tag{3.4}$$

The first few generalized Gaussian Adrien numbers with positive subscript and negative subscript are presented in the following Table 1.

Table 1. The first few generalized Gaussian Adrien numbers with positive subscript

n	GW_n
0	$W_0 + i(3W_2 - W_1 + -W_3)$
1	$W_1 + iW_0$
2	$W_2 + iW_1$
3	$W_3 + iW_2$
4	$3W_3 - W_2 - W_0 + iW_3$
5	$8W_3 - W_1 - 3W_2 - 3W_0 + i(3W_3 - W_2 - W_0 + iW_3)$

and with a negative subscript shown in Table 2.

We can define two special cases of $GW_n : GW_n(0, 1, 3+i, 8+3i) = GA_n$ is the sequence of Gaussian Adrien numbers , $GW_n : (4, 3+4i, 7+3i, 18+7i) = GB_n$ is the sequence of Gaussian Adrien-Lucas numbers.

Table 2. The first few generalized Gaussian Adrien numbers with negative subscript

n	GW_{-n}
0	$W_3 + i(3W_2 - W_1 - W_3)$
1	$3W_2 - W_1 - W_3 + i(3W_1 - W_0 - W_2)$
2	$3W_3 + W_0 - W_2 + i(3W_0 - 3W_2 + W_3)$
3	$3W_0 - 3W_2 + W_3 + i(10W_2 - 6W_1 - 3W_3)$
4	$10W_0 - 6W_1 - 3W_3 + i(10W_1 - 6W_0 - 3W_2)$
5	$10W_1 - 6W_0 - 3W_2 + i(10W_0 + 3W_1 - 18W_2 + 6W_3)$

So Gaussian Adrien numbers are defined by

$$GA_n = 3GA_{n-1} - GA_{n-2} - GA_{n-4}, \quad (3.5)$$

with the initial conditions

$$GA_0 = 0, GA_1 = 1, GA_2 = 3 + i, GA_3 = 8 + 3i.$$

Gaussian Adrien-Lucas numbers are defined by

$$GB_n = 3GB_{n-1} - GB_{n-2} - GB_{n-4}, \quad (3.6)$$

with the initial conditions

$$GB_0 = 4 + 4i, GB_1 = 3 + 4i, GB_2 = 7 + 3i, GB_3 = 18 + 7i.$$

$$GA_n = A_n + iA_{n-1,}$$

$$GB_n = B_n + iB_{n-1}.$$

The first few values of Gaussian Adrien numbers, Gaussian Adrien-Lucas numbers, with positive and negative subscript are given in the Table 3.

Table 3. Special cases of Gaussian generalized Adrien numbers and Gaussian Adrien-Lucas numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8
GA_n	0	1	$3+i$	$8+3i$	$21+8i$	$54+21i$	$138+54i$	$352+138i$	$897+352i$
GA_{-n}	0	0	$-i$	-1	i	$1-3i$	-3	$6i$	$6-10i$
GB_n	4	$3+4i$	$7+3i$	$18+7i$	$43+18i$	$108+43i$	$274+108i$	$696+274i$	$1771+696i$
GB_{-n}	4	$-2i$	$-2+9i$	$9-2i$	$-2-15i$	$-15-2i$	31	$-74i$	$-74+108i$

Next, we describe the Binet's formula for the Gaussian generalized Adrien numbers.

The Binet's formula for the Gaussian generalized Adrien numbers is

$$\begin{aligned} GW_n &= \frac{(\alpha GW_3-\alpha(3-\alpha)GW_2+(-\alpha^2+(3-1)\alpha+1)GW_1-GW_0)\alpha^n}{4\alpha^2+3\alpha-1} \\ &+\frac{(G\beta W_3-\beta(3-\beta)GW_2+(-\beta^2+(3-1)\beta+1)GW_1-GW_0)\beta^n}{4\beta^2+3\beta-1} \\ &+\frac{(\gamma GW_3-\gamma(3-\gamma)GW_2+(-\gamma^2+(3-1)\gamma+1)GW_1-GW_0)\gamma^n}{3\gamma-2} \\ &+\frac{GW_3-2GW_2-GW_1-GW_0}{-3} \\ &+i(\frac{(\alpha GW_3-\alpha 3-\alpha)GW_2+(-\alpha^2+(3-1)\alpha+1)GW_1-GW_0)\alpha^{n-1}}{4\alpha^2+3\alpha-1} \\ &+\frac{(G\beta W_3-\beta(3-\beta)GW_2+(-\beta^2+(3-1)\beta+1)GW_1-GW_0)\beta^{n-1}}{4\beta^2+3\beta-1} \\ &+\frac{(\gamma W_3-\gamma(3-\gamma)GW_2+(-\gamma^2+(3-1)\gamma+1)GW_1-GW_0)\gamma^{n-1}}{4\gamma^2+3\gamma-1} \\ &+\frac{GW_3-2GW_2-GW_1-GW_0}{-3}). \end{aligned}$$

Proof. The proof follows from (2.1) and (3.3). □

The previous Theorem gives the following results.

Corollary 3.1. *For all integers n, we have following identities,*

(a) $GA_n = \frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1}+\frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1}+\frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1}-\frac{1}{3}+$
$i(\frac{(2\alpha^2+\alpha+1)\alpha^{n-1}}{4\alpha^2+3\alpha-1}+\frac{(2\beta^2+\beta+1)\beta^{n-1}}{4\beta^2+3\beta-1}+\frac{(2\gamma^2+\gamma+1)\gamma^{n-1}}{4\gamma^2+3\gamma-1}-\frac{1}{3}).$

(b) $GB_n = \alpha^n+\beta^n+\gamma^n+1+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1).$

The next Theorem presents the generating function of Gaussian generalized Adrien numbers.

Theorem 3.2. *Let $f_{GW_n}(x)=\sum\limits_{n=0}^{\infty} GW_n x^n$ donate the generating function of Gaussian generalized Adrien numbers is given as follows:*

$$\begin{aligned} f_{GW_n}(z) &= \sum_{n=0}^{\infty} GW_n x^n \\ &= \frac{GW_0+(GW_1-3GW_0)x+(GW_2-3GW_1+GW_0)x^2+(GW_3-3GW_2+GW_1)x^3}{1-3x+x^2+x^4}. \quad (3.7) \end{aligned}$$

Proof. Using the definition of Gaussian Adrien numbers, and substracting $xf(x)$, $x^2f(x)$ and $x^3f(x)$ from $f(x)$

we obtain $(1 + x^2 - 3x^3 + x^4)f_{GW_n}(x)$

$$\begin{aligned}
(1-3x+x^2+x^4)f_{GW_n}(x) &= \sum_{n=0}^{\infty} GW_n x^n - 3x\sum_{n=0}^{\infty} GW_n x^n + x^2\sum_{n=0}^{\infty} GW_n x^n + x^4\sum_{n=0}^{\infty} GW_n x^n, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=0}^{\infty} GW_n x^{n+1} + \sum_{n=0}^{\infty} GW_n x^{n+2} + \sum_{n=0}^{\infty} GW_n x^{n+4}, \\
&= \sum_{n=0}^{\infty} GW_n x^n - 3\sum_{n=1}^{\infty} GW_{(n-1)} x^n + \sum_{n=2}^{\infty} GW_{(n-2)} x^n + \sum_{n=4}^{\infty} GW_{(n-4)} x^n, \\
&= (GW_0 + GW_1 x + GW_2 x^2 + GW_3 x^3) - 3(GW_0 x + GW_1 x^2 + GW_2 x^3) \\
&\quad +3(GW_0 x^2 + GW_1 x^3) + \sum_{n=4}^{\infty}(GW_n - 3GW_{n-1} + GW_{n-2} + GW_{n-4})x^n, \\
&= GW_0 + (GW_1 - 3GW_0)x + (GW_2 - 3GW_1 + 3GW_0)x^2 \\
&\quad +(GW_3 - 3GW_2 + GW_1)x^3.
\end{aligned}$$

and modifying above equation, we get (3.2). □

Corollary 3.3. *For all integers n, we have following identities:*

(a) $f_{GA_n}(x) = \sum\limits_{n=0}^{\infty} GA_n x^n = \dfrac{ix^2 + x}{x^4 + 2x^3 + 3x^2 - 7x + 1}$,

(b) $f_{GB_n}(x) = \sum\limits_{n=0}^{\infty} GB_n x^n = \dfrac{2ix^3 + (2-9i)\,x^2 - (9-4i)\,x + 4}{x^4 + 2x^3 + 3x^2 - 7x + 1}$.

Theorem (3.2) gives the following results as special cases,

$(1-3x+x^2+x^4)f_{GA_n}(x) = GA_0 + (GA_1 - 3GA_0)x + (GA_2 - 3GA_1 + 3GA_0)x^2 + (GA_3 - 3GA_2 + GA_1)x^3$
$= ix^2 + x, (1 - 3x + x^2 + x^4),$
$f_{GB_n}(x) = GB_0 + (GB_1 - 3GB_0)x + (GB_2 - 3GB_1 + GB_0)x^2 + (GB_3 - 3GB_2 + GB_1)x^3$
$= 2ix^3 + (2-9i)\,x^2 - (9-4i)\,x + 4.$

Lemma 3.4. *Suppose that $f_{GW_n}(x) = \sum\limits_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is the exponential Gaussian generating function of the generalized Adrien sequence $\{GW_n\}$.*

Then $\sum\limits_{n=0}^{\infty} GW_n \frac{x^n}{n!}$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} \\ &+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x}. \\ &+ i(\frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{\alpha(4\alpha^2 + 3\alpha - 1)} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{\beta(4\beta^2 + 3\beta - 1)} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{\gamma(4\gamma^2 + 3\gamma - 1)} e^{\gamma x} \\ &+ (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x})\end{aligned}$$

Proof. The proof follows from the Binet's formula of GW_n and $GW_n = W_n + iW_{n-1}$ Lemma(2.4).

$$\begin{aligned}\sum_{n=0}^{\infty} GW_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} (W_n + iW_{n-1}) \frac{x^n}{n!} = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} iW_{n-1} \frac{x^n}{n!} \\ & \frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - GW_0)}{4\alpha^2 + 3\alpha - 1} e^{\alpha x} \\ &+ \frac{(GW_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{4\beta^2 + 3\beta - 1} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - W_0)}{4\gamma^2 + 3\gamma - 1} e^{\gamma x} \\ &+ (\frac{W_3 - 2W_2 - W_1 - GW_0}{-3}) e^{x}. \\ &+ i(\frac{(\alpha W_3 - \alpha(3-\alpha)W_2 + (-\alpha^2 + (3-1)\alpha + 1)W_1 - W_0)}{\alpha(4\alpha^2 + 3\alpha - 1)} e^{\alpha x} \\ &+ \frac{(\beta W_3 - \beta(3-\beta)W_2 + (-\beta^2 + (3-1)\beta + 1)W_1 - W_0)}{\beta(4\beta^2 + 3\beta - 1)} e^{\beta x} \\ &+ \frac{(\gamma W_3 - \gamma(3-\gamma)W_2 + (-\gamma^2 + (3-1)\gamma + 1)W_1 - GW_0)}{\gamma(4\gamma^2 + 3\gamma - 1)} e^{\gamma x} + (\frac{W_3 - 2W_2 - W_1 - W_0}{-3}) e^{x})\end{aligned}$$

The previous Lemma 3.4 gives the following results as particular examples.

Corollary 3.5. *Exponential Gaussian generating function of Adrien and Adrien-Lucas numbers*

a) $\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} ((\frac{(2\alpha^2+\alpha+1)\alpha^n}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^n}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3})+$
$i(\frac{(2\alpha^2+\alpha+1)\alpha^{n-1}}{4\alpha^2+3\alpha-1} + \frac{(2\beta^2+\beta+1)\beta^{n-1}}{4\beta^2+3\beta-1} + \frac{(2\gamma^2+\gamma+1)\gamma^{n-1}}{4\gamma^2+3\gamma-1} - \frac{1}{3}))\frac{x^n}{n!}.$
$= (\frac{(2\alpha^2+\alpha+1)}{4\alpha^2+3\alpha-1} e^{\alpha x} + \frac{(2\beta^2+\beta+1)}{4\beta^2+3\beta-1} e^{\beta x} + \frac{(2\gamma^2+\gamma+1)\gamma^n}{4\gamma^2+3\gamma-1} - \frac{1}{3} e^{x})+$

$i(\frac{(2\alpha^2+\alpha+1)}{\alpha(4\alpha^2+3\alpha-1)}e^{\alpha x}+\frac{(2\beta^2+\beta+1)}{\beta(4\beta^2+3\beta-1)}e^{\beta x}+\frac{(2\gamma^2+\gamma+1)}{\gamma(4\gamma^2+3\gamma-1)}e^{\gamma x}-\frac{1}{3}e^{x}).$

b) $\sum\limits_{n=0}^{\infty}B_n\frac{x^n}{n!}=\sum\limits_{n=0}^{\infty}(\alpha^n+\beta^n+\gamma^n+1+i(\alpha^{n-1}+\beta^{n-1}+\gamma^{n-1}+1))\frac{x^n}{n!}=e^{\alpha x}+e^{\beta x}+e^{\gamma x}+e^{x}+i(\frac{1}{\alpha}e^{\alpha x}+\frac{1}{\beta}e^{\beta x}+\frac{1}{\gamma}e^{\gamma x}+e^{x}).$

4 OBTAINING BINET FORMULA FROM GENERATING FUNCTION

We next find Binet formula generalized Gaussian Adrien number $\{GW_n\}$ by the use of generating function for GW_n.

Theorem 4.1. *(Binet formula of generalized Gaussian Adrien numbers)*

$$GW_n=\frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \quad (4.1)$$

where

$$\begin{aligned}
q_1 &= GW_0\alpha^3+(GW_1-3GW_0)\alpha^2+(GW_2+GW_1+GW_0)\alpha+(GW_3+GW_2+GW_1),\\
q_2 &= GW_0\beta^3+(GW_1-3GW_0)\beta^2+(GW_2+GW_1+GW_0)\beta+(GW_3+GW_2+GW_1),\\
q_3 &= GW_0\gamma^3+(GW_1-3GW_0)\gamma^2+(GW_2+GW_1+GW_0)\gamma+(GW_3+GW_2+GW_1),\\
q_4 &= GW_0\delta^3+(GW_1-3GW_0)\delta^2+(GW_2+GW_1+GW_0)\delta+(GW_3+GW_2+GW_1).
\end{aligned}$$

Proof. Let

$$h(x)=1-3x+x^2+x^4.$$

Then for some α,β,γ and δ we write

$$h(x)=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x),$$

i.e.,

$$1-3x+x^2=(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x), \quad (4.2)$$

Hence $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x})=1-\frac{3}{x}+\frac{1}{x^2}+\frac{1}{x^4}=0.$$

This implies $x^4-3x^3+x^2+u=0$. Now, by it follows that

$$\sum_{n=0}^{\infty}GW_nx^n=\frac{GW_0+(GW_1-3GW_0)x+(GW_2-3GW_1+GW_0)x^2+(GW_3-3GW_2+GW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\begin{aligned}
&\frac{GW_0+(GW_1-3GW_0)x+(GW_2-3GW_1+GW_0)x^2+(GW_3-3GW_2+GW_1)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}\\
=\;&\frac{B_1}{(1-\alpha x)}+\frac{B_2}{(1-\beta x)}+\frac{B_3}{(1-\gamma x)}+\frac{B_4}{(1-\delta x)}.
\end{aligned} \quad (4.3)$$

So

$$\begin{aligned}&GW_0+(GW_1-3GW_0)x+(GW_2-3GW_1+GW_0)x^2+(GW_3-3GW_2+GW_1)x^3\\=\;&B_1(1-\beta x)(1-\gamma x)(1-\delta x)+B_2(1-\alpha x)(1-\gamma x)(1-\delta x)\\&+B_3(1-\alpha x)(1-\beta x)(1-\delta x)+B_3(1-\alpha x)(1-\beta x)(1-\gamma x).\end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $GW_0+(GW_1-3GW_0)\frac{1}{\alpha}+(GW_2-3GW_1+GW_0)\frac{1}{\alpha^2}+(GW_3-3GW_2+GW_1)\frac{1}{\alpha^3}$ $=B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.

This gives

$$\begin{aligned}B_1&=\frac{\alpha^3(GW_0+(GW_1-3GW_0)\frac{1}{\alpha}+(GW_2-3GW_1+GW_0)\frac{1}{\alpha^2}+(GW_3-6GW_2+GW_1)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\\&=\frac{GW_0\alpha^3+(GW_1-GW_0)\alpha^2+(GW_2-3GW_1+GW_0)\alpha+(GW_3-3GW_2+GW_1)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}B_2&=\frac{GW_0\beta^3+(GW_1-3GW_0)\beta^2+(GW_2-3GW_1+GW_0)\beta+(GW_3-3GW_2+GW_1)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},\\B_3&=\frac{GW_0\gamma^3+(GW_1-3GW_0)\gamma^2+(GW_2-3GW_1+GW_0)\gamma+(GW_3-3GW_2+GW_1)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},\\B_4&=\frac{GW_0\delta^3+(GW_1-3GW_0)\delta^2+(GW_2-3GW_1+GW_0)\delta+(GW_3-3GW_2+GW_1)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.\end{aligned}$$

Thus (4.3)can be written as

$$\sum_{n=}^{\infty}GW_nx^n=B_1(1-\alpha x)^{-1}+B_2(1-\beta x)^{-1}+B_3(1-\gamma x)^{-1}+B_4(1-\delta x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty}GW_nx^n&=B_1\sum_{n=0}^{\infty}\alpha^nx^n+B_2\sum_{n=0}^{\infty}\beta^nx^n+B_3\sum_{n=0}^{\infty}\gamma^nx^n+B_4\sum_{n=0}^{\infty}\delta^nx^n\\&=\sum_{n=0}^{\infty}(B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n)x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$GW=B_1\alpha^n+B_2\beta^n+B_3\gamma^n+B_4\delta^n.$$

and then we get (4.1). □

5 SOME IDENTITIES ABOUT RECURRENCE RELATIONS OF GAUSSIAN GENERALIZED ADRIEN NUMBERS

In this section, we present some identities on Gaussian Adrien, *Gaussian Adrien-Lucas.*

Theorem 5.1. *The following equations hold for all integer* n

$$\begin{aligned} GA_n &= \frac{34}{261}GB_{n+3} - \frac{50}{261}GB_{n+2} - \frac{22}{261}GB_{n+1} - \frac{49}{261}GB_n, \\ GB_n &= -GA_{n+3} + 3GA_{n+2} + GA_{n+1} - 4GA_n. \end{aligned} \tag{5.1}$$

Proof. To proof identity (5.1), we can write

$$GA_n = aGB_{n+3} + bGB_{n+2} + cGB_{n+1} + dGB_n.$$

Solving the system of equations

$$\begin{aligned} GA_0 &= aGB_3 + bGB_2 + cGB_1 + dGB_0, \\ GA_1 &= aGB_4 + bGB_3 + cGB_2 + dGB_1, \\ GA_2 &= aGB_5 + bGB_4 + cGB_3 + dGB_2, \\ GA_3 &= aGB_6 + bGB_5 + cGB_4 + dGB_3. \end{aligned}$$

we get $a = \frac{34}{261}, b = -\frac{50}{261}, c = -\frac{22}{261}, d = -\frac{49}{261}$.The other identities can be found similarly. □

Lemma 5.2. *(Frontczak, 2018) Let's assume that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *is the generating function of the sequence* $\{a_n\}_{n\geq 0}$*. Then the generating functions of the sequences* $\{a_{2n}\}_{n\geq 0}$ *and* $\{a_{2n+1}\}_{n\geq 0}$ *are stated as*

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}.$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}.$$

respectively.

The generating functions of the even and odd-indexed Gaussian generalized Adrien sequences are provided by the following theorem.

Theorem 5.3. *The generating functions of the sequence* GW_{2n} *and* GW_{2n+1} *are provided by*

$$f_{GW_{2n}}(x) = \frac{x^3(GW_2 - 3GW_1 + GW_0) + x^2(3GW_3 - 8GW_2 + 2GW_0) + x(GW_2 - 7GW_0) + GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}, \tag{5.2}$$

$$f_{GW_{2n+1}}(x) = \frac{x^3(GW_3 - 3GW_2 + GW_1) + x^2\left(GW_3 - 3GW_2 + 2GW_1 - 3GW_0\right) + x\left(GW_3 - 7GW_1\right) + GW_1}{x^4 + 2x^3 + 3x^2 - 7x + 1}. \tag{5.3}$$

Proof. We only proof (5.2). From Theorem (3.2) we can obtain following identities:

$$f_{GW_n}(\sqrt{x}) = \frac{(GW_2 - 3GW_1 + GW_0)\sqrt{x^3} + (3GW_3 - 8GW_2 + 2GW_0)\sqrt{x^2} + (GW_2 - 7GW_0)\sqrt{x} + GW_0}{x^2 + 2\sqrt{x^3} + 3x - 7\sqrt{x} + 1},$$

$$f_{GW_n}(-\sqrt{x}) = -\frac{(GW_3 - 3GW_2 + GW_1)\sqrt{x^3} + (GW_3 - 3GW_2 + 2GW_1 - 3GW_0)\sqrt{x^2} + (GW_3 + 7GW_1)\sqrt{x} - GW_1}{x^2 + 2\sqrt{x^3} + 3x - 7\sqrt{x} + 1}.$$

Thus, the result follows from Lemma (5.2) can be proved . The other identity can be found similarly. □

From Theorem (5.3), we get the following Corollary.

Corollary 5.4.

$$\begin{aligned}
f_{GA_{2n}}(x) &= \frac{ix^3 + ix^2 + (3+i)\,x}{x^4 + 2x^3 + 3x^2 - 7x + 1}, \\
f_{GA_{2n+1}}(x) &= \frac{x^2 + (1+3i)\,x + 1}{x^4 + 2x^3 + 3x^2 - 7x + 1}, \\
f_{GB_{2n}}(x) &= \frac{(2-9i)\,x^3 + (6-3i)\,x^2 - (21-3i)\,x + 4}{x^4 + 2x^3 + 3x^2 - 7x + 1}, \\
f_{GB_{2n+1}}(x) &= \frac{2ix^3 - (9-6i)\,x^2 - (3+21i)\,x + (3+4i)}{x^4 + 2x^3 + 3x^2 - 7x + 1}.
\end{aligned}$$

From Corollary (5.4) we can obtain the following corollary which presents the identities on Gaussian Adrien sequences.

Corollary 5.5. **(a)** $(3+i)GB_{2n-2} + iGB_{2n-4} + iGB_{2n-6} = 4GA_{2n} - (21-3i)GA_{2n-2}$
$+(6-3i)GA_{2n-4} + (2-9i)GA_{2n-6},$

(b) $GB_{2n} + (1+3i)\,GB_{2n-2} + GB_{2n-4} = 4GA_{2n+1} - (21-3i)GA_{2n-1} + (6-3i)GA_{2n-3}$
$+(2-9i)GA_{2n-5},$

(c) $(3+4i)GB_{2n} - (3+21i)GB_{2n-2} - (9-6i)GB_{2n-4} + (2i)GB_{2n-6} = 4GB_{2n+1} - (21-3i)GB_{2n-1}$
$+(6-3i)GB_{2n-3} + (2-9i)GB_{2n-5},$

d) $GB_{2n+1} + (1+3i)GB_{2n-1} + GB_{2n-3} = (3+4i)GA_{2n+1} - (3+21i)GA_{2n-1} - (9-6i)GA_{2n-3}$
$+(2i)GA_{2n-5},$

(e) $(3+i)GB_{2n-1} + iGB_{2n-3} + iGB_{2n-5} = (3+4i)GA_{2n} - (3+21i)GA_{2n-2} - (9-6i)GA_{2n-4}$
$+(2i)GA_{2n-6},$

(f) $GA_{2n} + (1+3i)GA_{2n-2} + GA_{2n-4} = (3+i)GA_{2n-1} + iGA_{2n-3} + iGA_{2n-5}.$

Proof. From Corollary (5.4) we obtain

$$(ix^3 + ix^2 + (3+i)\,x)f_{GA_{2n}}(x) = ((2-9i)\,x^3 + (6-3i)\,x^2 - (21-3i)x + 4)f_{GB_{2n}}(x).$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ix^3+ix^2+(3+i)\,x\sum_{n=0}^{\infty}GB_{2n}x^n, \\
&= (3+i)\,x\sum_{n=0}^{\infty}GB_{2n}x^n+ix^2\sum_{n=0}^{\infty}GB_{2n}x^n+ix^3\sum_{n=0}^{\infty}GB_{2n}x^n, \\
&= (3+i)\sum_{n=0}^{\infty}GB_{2n}x^{n+1}+i\sum_{n=0}^{\infty}GB_{2n}x^{n+2}+i\sum_{n=0}^{\infty}GB_{2n}x^{n+3}, \\
&= (3+i)\sum_{n=1}^{\infty}GB_{2n-2}x^n+i\sum_{n=2}^{\infty}GB_{2n-4}x^n+i\sum_{n=3}^{\infty}GB_{2n-6}x^n, \\
&= (12+4i)x+(18+20i)x^2+\sum_{n=3}^{\infty}((3+i)\,GB_{2n-2}+iGB_{2n-4}+iGB_{2n-6})x^n.
\end{aligned}$$

whereas the RHS (right hand side) is equal to

$$\begin{aligned}
RHS &= ((2-9i)\,x^3+(6-3i)\,x^2-(21-3i)x+4)\sum_{n=0}^{\infty}GA_{2n}x^n \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)x\sum_{n=0}^{\infty}GA_{2n}x^n+(6-3i)\,x^2\sum_{n=0}^{\infty}GA_{2n}x^n+(2-9i)\,x^3\sum_{n=0}^{\infty}GA_{2n}x^n \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)\sum_{n=0}^{\infty}GA_{2n}x^{n+1}+(6-3i)\sum_{n=0}^{)\infty}GA_{2n}x^{n+2}+(2-9i)\sum_{n=0}^{\infty}GA_{2n}x^{n+3} \\
&= 4\sum_{n=0}^{\infty}GA_{2n}x^n-(21-3i)\sum_{n=1}^{\infty}GA_{2n-2}x^n+(6-3i)\sum_{n=2}^{\infty}GA_{2n-4}x^n+(2-9i)\sum_{n=3}^{\infty}GA_{2n-6}x^n \\
&\quad (12+4i)x+(18+20i)x^2+\sum_{n=3}^{\infty}(4GA_{2n}-(21-3i)GA_{2n-2}+(6-3i)GA_{2n-4}+(2-9i)GA_{2n-6})x^n
\end{aligned}$$

Comparing the coefficients and the proof of the first identity (a) is done. We can show other identity similarly. □

We can get an identitiy related to Gaussian Genaralized Adrien numbers given below.

Theorem 5.6. *For all integers m, n the following identities hold:*

$$GW_{m+n}=A_{m-2}GW_{n+3}+(-A_{m-3}-A_{m-5})GW_{n+2}+(-A_{m-4})GW_{n+1}-A_{m-3}GW_n.$$

Proof. First we assume that $m, n\geq 0$ then (5.6) can be proved by mathematical induction on m. If $m=0$ we get

$$GW_n=A_{-2}GW_{n+3}+(-A_{-3}-A_{-5})GW_{n+2}+(-A_{-4})GW_{n+1}-A_{-3}GW_n.$$

which is true since $A_{-2}=0,\ A_{-3}=-1,\ A_{-4}=0,\ A_{-5}=1$. Assume that the equality holds for $m\leq k$. For $m=k+1$, we get

$GW_{k+1+n} = 3GW_{n+k} - GW_{n+k-1} - GW_{n+k-3},$
$=3A_{k-2}GW_{n+3} + (-A_{k-3} - A_{k-5})GW_{n+2}$
$+3(-A_{k-4})GW_{n+1} - A_{k-3}GW_n$
$-(A_{k-3}GW_{n+3} + (-A_{k-4} - A_{k-6})GW_{n+2} + (-A_{-5})GW_{n+1} - A_{k-4}GW_n)$
$-A_{k-5}GW_{n+3} + (-A_{k-6} - A_{k-8})GW_{n+2} + (-A_{k-6})GW_{n+1} - A_{k-6}GW_n.$

Consequently, by mathematical induction on m, this proves Theorem (5.6).

The other cases of m, n can be proved smilarly for all integers m, n. □

Taking $GW_n = GA_n$ or $GW_n = GB_n$ in above Theorem, respectively, we get:

Corollary 5.7.

$$\begin{aligned} GA_{m+n} &= A_{m-2}GA_{n+3} + (-A_{m-3} - A_{m-5})GA_{n+2} + (-A_{m-4})GA_{n+1} - A_{m-3}GA_n, \\ GB_{m+n} &= A_{m-2}GB_{n+3} + (-A_{m-3} - A_{m-5})GB_{n+2} + (-A_{m-4})GK_{n+1} - A_{m-3}GB_n. \end{aligned}$$

6 SIMSON'S FORMULA

In this section, we present Simson's formula of generalized Gaussian Adrien numbers. This is a special case of [(Soykan, 2019c),Theorem 4.1].

Theorem 6.1. *For all integers n, we can write the following equality:*

$$\begin{vmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1} & GW_{n-2} & GW_{n-3} \end{vmatrix} = \begin{vmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{vmatrix}$$
$$= (GW_0 + GW_1 + 2GW_2 - GW_3)(-GW_3^3 + 5GW_2^3 + GW_1^3 + GW_0^3 - (GW_0 + 3GW_1 - 7GW_2)GW_3^2$$
$$+(3GW_0 - 4GW_1 - 14GW_3)GW_2^2 + (2GW_0 + GW_2 - 6GW_3)GW_1^2 - (GW_1 + 2GW_3)GW_0^2$$
$$+13GW_1GW_2GW_3 + GW_0GW_2GW_3 + 5GW_0GW_1GW_3 - 7GW_0GW_1GW_2).$$

Proof. Using Theorem (3) it can be proved by using induction use [(Soykan, 2019c),Theorem 4.1] From the Theorem (6.1) we get the following Corollary.

Corollary 6.2. *For all integers n, the Simson's formulas of Adrien and Adrien Lucas numbers are given as respectively.*

(a) $\begin{vmatrix} GA_{n+3} & GA_{n+2} & GA_{n+1} & GA_n \\ GA_{n+2} & GA_{n+1} & GA_n & GA_{n-1} \\ GA_{n+1} & GA_n & GA_{n-1} & GA_{n-2} \\ GA_n & GA_{n-1} & GA_{n-2} & GA_{n-3} \end{vmatrix} = 1 - 3i.$

(b) $\begin{vmatrix} GB_{n+3} & GB_{n+2} & GB_{n+1} & GB_n \\ GB_{n+2} & GB_{n+1} & GB_n & GB_{n-1} \\ GB_{n+1} & GB_n & GB_{n-1} & GB_{n-2} \\ GB_n & GB_{n-1} & GSB_{n-2} & GB_{n-3} \end{vmatrix} = -783 + 2349i.$

7 SUM FORMULAS

In this section, we identify some sum formulas of generalized Gaussian Adrien numbers.

Theorem 7.1. *For all integers $n \geq 0,$ we get sum formulas below:*

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{3}(-(n+3)GW_{n+3} + (2n+7)GW_{n+2} + (n+2)GW_{n+1} + (n+4)GW_n$
$+3GW_3 - 7GW_2 - 2GW_1 - GW_0).$

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{3}(-(n+2)GW_{2n+2} + (2n+5)GW_{2n+1} + (n+3)GW_{2n} + (n+2)GW_{2n-1}$
$+2GW_3 - 4GW_2 - 3GW_1).$

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(-(n+1)GW_{2n+2} + (2n+5)GW_{2n+1} + (n+2)GW_{2n} + (n+2)GW_{2n-1}$
$+2GW_3 - 5GW_2 - 2GW_0).$

Proof. It can be proved by using Theorem 3.10 in Soykan (Soykan, 2025). □

As a special case of the Theorem 7.1, we present following Corollary.

Corollary 7.2. *For all integers $n \geq 0,$ we get sum formulas below:*

(a) $\sum_{k=0}^{n} GA_k = \frac{1}{3}(-(n+3)GA_{n+3} + (2n+7)GA_{n+2} + (n+2)GA_{n+1} + (n+4)GA_n + 1 + 2i).$

(b) $\sum_{k=0}^{n} GA_{2k} = \frac{1}{3}(-(n+2)GA_{2n+2} + (2n+5)GA_{2n+1} + (n+3)GA_{2n} + (n+2)GA_{2n-1} + 1 + 2i).$

(c) $\sum_{k=0}^{n} GA_{2k+1} = \frac{1}{3}(-(n+1)GA_{2n+2} + (2n+5)GA_{2n+1} + (n+2)GA_{2n} + (n+2)GA_{2n-1} + 1 + i).$

As a special case of the Theorem 7.1, we present following Corollary.

Corollary 7.3. *For all integers $n \geq 0,$ we get sum formulas below:*

(a) $\sum_{k=0}^{n} GB_k = \frac{1}{3}(-(n+3)GB_{n+3} + (2n+7)GB_{n+2} + (n+2)GB_{n+1} + (n+4)GB_n - 5 - 8i).$

(b) $\sum_{k=0}^{n} GB_{2k} = \frac{1}{3}(-(n+2)GB_{2n+2} + (2n+5)GB_{2n+1} + (n+3)GB_{2n} + (n+2)GB_{2n-1} - 1 - 10i.$

(c) $\sum_{k=0}^{n} GB_{2k+1} = \frac{1}{3}(-(n+1)GB_{2n+2} + (2n+5)GB_{2n+1} + (n+2)GB_{2n} + (n+2)GB_{2n-1} - 7 - i).$

Next, we give the ordinary generating functions of some special cases of Gaussian generalized Adrien numbers.

Theorem 7.4. *The ordinary generating functions of the sequences W_{2n}, W_{2n+1} are given as follows:*

(a) $\sum_{n=0}^{\infty} GW_{2n}x^n = \dfrac{(3x^2)GW_3 + (x^3 - 8x^2 + x)GW_2 + (-3x^3)GW_1 + (x^3 + 2x^2 - 7x + 1)GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

(b) $\sum_{n=0}^{\infty} GW_{2n+1}x^n = \dfrac{(x^3 + x^2 + x)GW_3 - (3x^3 + 3x^2)GW_2 + (x^3 + 2x^2 - 7x + 1)GW_1 + (-3x^2)GW_0}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Adrien numbers (Take $W_n = GA_n$ whit $GA_0 = 0, GA_1 = 1, GA_2 = 3 + i, GA_3 = 8 + 3i.$)

Corollary 7.5. *For $n \geq 0$ Gaussian Adrien numbers have the following properties:*

(a) $\sum_{n=0}^{\infty} GA_{2n}x^n = \dfrac{ix^3 + ix^2 + (3+i)\,x}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

(b) $\sum_{n=0}^{\infty} GA_{2n+1}x^n = \dfrac{x^2 + (1+3i)\,x + 1}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Adrien-Lucas numbers (Take $W_n = Gb_n$ whit $GB_0 = 4, GB_1 = 3+4i, GB_2 = 7+3i, GB_3 = 18+7i.$)

Corollary 7.6. *For $n \geq 0$ Gaussian Adrien-Lucas numbers have the following properties:*

(a) $\sum_{n=0}^{\infty} GB_{2n}x^n = \dfrac{(2-9i)\,x^3 + (6-3i)\,x^2 - (21-3i)\,x + 4}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

(b) $\sum_{n=0}^{\infty} GB_{2n+1}x^n == \dfrac{2ix^3 - (9-6i)\,x^2 - (3+21i)\,x + 3 + 4i}{x^4 + 2x^3 + 3x^2 - 7x + 1}.$

8 MATRIX FORMULATION OF $\mathbf{GW}_n$

In this section, we review the matrix representation of generalized Gaussian Adrien numbers.

We define the square matrix M of order 4 as

$$M = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $detM = 1$. Note that

$$M^n = \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix}.$$

for the proof see (Soykan, 2021e).

Then we give the following lemma.

Lemma 8.1. *For $n \geq 0$ the following identitiy is true:*

$$\begin{pmatrix} GW_{n+3} \\ GW_{n+2} \\ GW_{n+1} \\ GW_n \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

Proof. The identitiy(8.1) can be proved by mathematical induction on n. If $n = 0$ we obtain

$$\begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix},$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identitiy is true

$$\begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} &= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} GW_3 \\ GW_2 \\ GW_1 \\ GW_0 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \\ GW_k \end{pmatrix} \\
&= \begin{pmatrix} GW_{k+4} \\ GW_{k+3} \\ GW_{k+2} \\ GW_{k+1} \end{pmatrix}.
\end{aligned}$$

Consequently, by mathematical induction on n, the proof completed. □

We define

$$N_{Gw} = \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix}, \tag{8.1}$$

$$E_{Gw} = \begin{pmatrix} GW_{n+3} & GW_{n+2} & GW_{n+1} & GW_n \\ GW_{n+2} & GW_{n+1} & GW_n & GW_{n-1} \\ GW_{n+1} & GW_n & GW_{n-1} & GW_{n-2} \\ GW_n & GW_{n-1-} & GW_{n-2} & GW_{n-3} \end{pmatrix}. \tag{8.2}$$

Now, we have the following theorem with N_{Gw} and E_{Gw}

Theorem 8.2. *Using N_{Gw} and E_{Gw} , we get*

$$A^n N_{Gw} = E_{Gw}.$$

Proof. Note that we get

$$\begin{aligned}
A^n N_{Gw} &= \begin{pmatrix} A_{n+1} & -A_n - A_{n-2} & -A_{n-1} & -A_n \\ A_n & -A_{n-1} - A_{n-3} & -A_{n-2} & -A_{n-1} \\ A_{n-1} & -A_{n-2} - A_{n-4} & -A_{n-3} & -A_{n-2} \\ A_{n-2} & -A_{n-3} - A_{n-5} & -A_{n-4} & -A_{n-3} \end{pmatrix} \begin{pmatrix} GW_3 & GW_2 & GW_1 & GW_0 \\ GW_2 & GW_1 & GW_0 & GW_{-1} \\ GW_1 & GW_0 & GW_{-1} & GW_{-2} \\ GW_0 & GW_{-1} & GW_{-2} & GW_{-3} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.
\end{aligned}$$

where

$$
\begin{aligned}
a_{11} &= A_{n+1}GW_3 + (-A_n - A_{n-2})GW_2 + (-A_{n-1})GW_1 + (-A_n)GW_0, \\
a_{12} &= A_{n+1}GW_2 + (-A_n - A_{n-2})GW_1 + (-A_{n-1})GW_0 + (-A_n)GW_{-1}, \\
a_{13} &= A_{n+1}GW_1 + (-A_n - A_{n-2})GW_0 + (-A_{n-1})GW_{-1} + (-A_n)GW_{-2}, \\
a_{14} &= A_{n+1}GW_0 + (-A_n - A_{n-2})GW_{-1} + (-A_{n-1})GW_{-2} + (-A_n)GW_{-3}, \\
a_{21} &= A_nGW_3 + (-A_{n-1} - A_{n-3})GW_2 + (-A_{n-2})GW_1 + (-A_{n-1})GW_0, \\
a_{22} &= A_nGW_2 + (-A_{n-1} - A_{n-3})GW_1 + (-A_{n-2})GW_0 + (-A_{n-1})GW_{-1}, \\
a_{23} &= A_nGW_1 + (-A_{n-1} - A_{n-3})GW_0 + (-A_{n-2})GW_{-1} + (-A_{n-1})GW_{-2}, \\
a_{24} &= A_nGW_0 + (-A_{n-1} - A_{n-3})GW_{-1} + (-A_{n-2})GW_{-2} + (-A_{n-1})GW_{-3}, \\
a_{31} &= A_{n-1}GW_3 + (-A_{n-2} - A_{n-4})GW_2 + (-A_{n-3})GW_1 + (-A_{n-2})GW_0, \\
a_{32} &= A_{n-1}GW_2 + (-A_{n-2} - A_{n-4})GW_1 + (-A_{n-3})GW_0 + (-A_{n-2})GW_{-1}, \\
a_{33} &= A_{n-1}GW_1 + (-A_{n-2} - A_{n-4})GW_0 + (-A_{n-3})GW_{-1} + (-A_{n-2})GW_{-2}, \\
a_{34} &= A_{n-1}GW_0 + (-A_{n-2} - A_{n-4})GW_{-1} + (-A_{n-3})GW_{-2} + (-A_{n-2})GW_{-3}, \\
a_{41} &= A_{n-2}GW_3 + (-A_{n-3} - A_{n-5})GW_2 + (-A_{n-4})GW_1 + (-A_{n-3})GW_0, \\
a_{42} &= A_{n-2}GW_2 + (-A_{n-3} - A_{n-5})GW_1 + (-A_{n-4})GW_0 + (-A_{n-3})GW_{-1}, \\
a_{43} &= A_{n-2}GW_1 + (-A_{n-3} - A_{n-5})GW_0 + (-A_{n-4})GW_{-1} + (-A_{n-3})GW_{-2}, \\
a_{44} &= A_{n-2}GW_0 + (-A_{n-3} - A_{n-5})GW_{-1} + (-A_{n-4})GW_{-2} + (-A_{n-3})GW_{-3}.
\end{aligned}
$$

Using the Theorem 5.6 the proof is done. □

By taking GW_n =GA_n with GA_0, GA_1, GA_2, GA_3 in (8.1) and (8.2)
GW_n =GB_n with GB_0, GB_1, GB_2, GB_3 in (8.1) and (8.2)

respectively, we get:

$$
\begin{aligned}
N_{GA} &= \begin{pmatrix} 8+3i & 3+i & 1 & 0 \\ 3+i & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_{GA} = \begin{pmatrix} GA_{n+3} & GA_{n+2} & GA_{n+1} & GA_0 \\ GA_{n+2} & GA_{n+1} & GA_n & GA_{n-1} \\ GA_{n+1} & GA_n & GA_{n-1} & GA_{n-2} \\ GA_n & GA_{n-1} & GA_{n-2} & GA_{n-3} \end{pmatrix}, \\
N_{GB} &= \begin{pmatrix} 18+7i & 7+3i & 3+4i & 4 \\ 7+3i & 3+4i & 4 & 4i \\ 3+4i & 4 & 4i & -2 \\ 4 & -4i & -2 & 9-2i \end{pmatrix},\ E_{GB} = \begin{pmatrix} GB_{n+3} & GB_{n+2} & GB_{n+1} & GB_0 \\ GB_{n+2} & GB_{n+1} & GB_n & GB_{n-1} \\ GB_{n+1} & GB_n & GB_{n-1} & GB_{n-2} \\ GB_n & GB_{n-1} & GB_{n-2} & GB_{n-3} \end{pmatrix}.
\end{aligned}
$$

From Theorem [8.2], we can write the following corollary.

Corollary 8.3. *The following identities are hold:*

(a) $A^n N_{GA} = E_{GA}$.

(b) $A^n N_{GB} = E_{GB}$.

9 CONCLUSIONS

Recurrence relations have been widely studied in the literature owing to their versatility and applicability across various fields, including physics, engineering, architecture, the natural sciences, and the arts. Among these, sequences defined by second-order recurrence relations—such as the Fibonacci, Lucas, Pell, and

Jacobsthal sequences—hold a particularly prominent place. The Fibonacci sequence, for instance, achieved historical significance through its application to the rabbit population model presented by Leonardo de Pisa in his 1202 work Liber Abaci. Both Fibonacci and Lucas sequences have inspired extensive research due to their elegant structural properties and numerous remarkable identities.

In this study, we introduce a class of fourth-order recurrence relations termed the Gaussian Generalized Adrien numbers, along with two notable special cases. We derive various structural properties of these sequences, including Binet-type formulas, ordinary and exponential generating functions, Simson-type identities, summation formulas, recurrence characteristics, and matrix representations.

Linear recurrence relations constitute powerful tools in both theoretical analysis and applied mathematics.

In the following lists, we explore illustrative applications of such relations, beginning with those governed by second-order structures.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see (Azak, 2022).
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see (Büyük and Taşkıra, 2022).
- For the application of Jacobsthal numbers to special matrices, see (Vasanthi and Sivakumar, 2022).
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see (Gül, 2022).
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see (Yılmaz, 2022b).
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see (Yılmaz, 2022a).
- For the applications of generalized Fibonacci numbers to binomial sums, see (Ulutaş and Toy, 2022).
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see (Soykan and Taşdemir, 2022).
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see (Soykan, 2021d).
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see (Deveci and Shannon, 2022).
- For the application of Generalized Fibonacci Matrices to Cryptography, see (Prasad and Mahato, 2022).
- For the application of higher order Jacobsthal numbers to quaternions, see (Özkan and Uysal, 2023).
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see (Goy and Shattuck, 2019).
- For the applications of Fibonacci numbers to lacunary statistical convergence, see (Bilgin, 2021).
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see (Kişi and Tuzcuoglu, 2020).
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see (Kişi and Debnathb, 2022).
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see (Kumari et al., 2023).
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see (Şentürk et al., 2022).
- For the application of special cases of Horadam numbers to Neutrosophic analysis see (Gökbaş et al., 2023).
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see (Daşdemir, 2021).

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see (Cerda-Morales, 2017a) and (Cerda-Morales, 2017b), respectively.
- For the application of Tribonacci numbers to special matrices, see [(Yilmaz and Taskara, 2014).

- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see (Shtayat and Al-Kateeb, 2019) and (Basu and Das, 2014), respectively.
- For the application of Pell-Padovan numbers to groups, see (Deveci and Shannon, 2017).
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see (Göcen, 2022).
- For the application of Gaussian Tribonacci numbers to various graphs, see (Sunitha and Sheriba, 2022).
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see (Dikmen and Altınsoy, 2022).For the application of Narayan numbers to finite groups see (Kuloğlu et al., 2022).
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see (Soykan et al., 2022a).
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see (Soykan et al., 2022b).
- For the application of generalized Tribonacci numbers to Gaussian numbers, see (Soykan et al., 2018).
- For the application of generalized Tribonacci numbers to Sedenions, see (Soykan et al., 2020a).
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see (Soykan, 2020a).
- For the application of generalized Tribonacci numbers to circulant matrix, see (Soykan, 2021b).
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see (Taşyurdu and Polat, 2021).
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see (Diskaya et al., 2023).

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see (Soykan, 2019d).
- For the application of generalized Tetranacci numbers to Gaussian numbers, see (Soykan, 2019a).
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see (Soykan, 2019b).
- For the application of generalized Tetranacci numbers to binomial transform, see (Soykan, 2021c).

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see (Sivakumar and James).
- For the application of generalized Pentanacci numbers to quaternions, see (Soykan et al., 2020b).
- For the application of generalized Pentanacci numbers to binomial transform, see (Soykan, 2021a).

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

COMPETING INTERESTS)

Authors have declared that no competing interests exist.

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