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Investigation of Selected Versions of Fourth Order Runge-Kutta Algorithms as Simulation Tools for Harmonically Excited Duffing Oscillator

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Authors' contributions

This study was carried out in collaboration between both authors. Authors TAOS and AIA designed the study. Author AIA carried out the literature searches, developed the program used and the first draft of the manuscript. Both authors managed the analysis of the study, read and approved the final manuscript.

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ABSTRACT

This study aim is to investigate the properties of selected fourth order Runge-Kutta algorithms. Fifty-five versions of fourth order Runge-Kutta (RK_1, RK_2, RK_3 …, RK_55) methods; inclusive of the classical fourth order RK version, were selected. Thereafter, these versions were used to simulate, with a constant and adaptive step-size algorithm, the dynamics of the harmonically excited Duffing Oscillator over a range of parameters and initial conditions. The simulation was carried out with a FORTRAN program developed and validated by comparing the program generated Poincaré section with literature standard. The number of successive steps taken between start and end of simulation periods was recorded for each simulation run. A total of 91809 simulations were run. The number of successive steps taken between start and end of simulation periods show that significant variations exists among different versions of the same Runge-Kutta order used for seeking solution of Duffing oscillator dynamics. Ranking results by the number of successive steps showed that RK_55 is not the fastest version available, despite its popularity, as other versions including RK_17, RK_2, RK_14, RK_20, and RK_8 outperformed it. Furthermore, the version performance was observed to be highly dependent on the excitation frequency, but not on initial conditions.

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Keywords: Runge-Kutta; duffing oscillator; adaptive; FORTRAN.

NOMENCLATURES

$\ddot{x}$	*:acceleration of the system*	h	*:step size*
$\dot{x}$	*:velocity of the system*	Er	*:local truncation error*
x	*:displacement of the system*	S_N	*:system number*
γ	*:damping constant*	ϕ	*:increment function*
α	*:linear stiffness coefficient*	s	*:number of stages of the RK method*
β	*:non-linear stiffness coefficient*	b_i	*:vector of the quadrature weights*
P_O	*:forcing amplitude*	k_i	*:vector of derivatives (slopes)*
ω	*:excitation (angular) frequency*	c_j	*:vector of nodes*
t	*:time*		
a_{jm}	*:matrix indicating dependence of the stages on the derivatives found at other stages*		

1. INTRODUCTION

Ordinary differential equations play a fundamental role in engineering as many physical phenomena are best formulated mathematically in terms of their rate of change. Solutions of these governing equations may be by Analytical (exact) or Numerical/Approximate methods. Ordinary differential equations may also be classified into linear and non-linear equations [1]. Non-linear science is a field of continuously growing interest to scientists, engineers and researchers, due to its usefulness in diverse fields such as Physics, Engineering, Biomedicine, etc. [2]. Numerical methods are of utmost importance to researchers as they can be used to solve non-linear model equations of interest, which generally do not possess analytical methods of solution [3].

The Duffing Oscillator (or Duffing Equation) is a non-linear second order ordinary differential equation named after Georg Duffing (1861-1944). Although George Duffing studied the free and forced harmonic vibration of an oscillator, which possessed quadratic and cubic non-linearity in the stiffness force, the term 'Duffing equation' is now used for any equation of an oscillator with a cubic stiffness term, with or without different types of damping or excitation [4]. This permits the equation to be used to model a wide range of linear and non-linear, physical and engineering systems. Consequently, the Duffing equation is one of the most intensively studied systems in dynamics. [2]. Examples of these systems include the equations of motion of plates, shells, woofers, optical fibers, prisms in fluid flow, flight motor of an insect, electrical circuit and pendulum. Another example of such systems is the vibration of beams [5]. In terms of solution of the Duffing equation, the exact solution of the undamped, unforced Duffing equation is well known in Literature [6], however, for other forms of the equation, approximate or numerical methods of solution are required. For example, Runge-Kutta (fourth order) algorithms were used by [2,7] to obtain bifurcation diagrams, very useful tools for the investigation of the dynamics, of the harmonically excited and forced Duffing oscillator.

Several numerical methods have been developed for the solutions of ordinary differential equations. Among these methods, the Runge-Kutta (RK) family of methods is distinguished, as evident in its widespread use and extensive research [1,8–11]. Runge-Kutta methods are a family of implicit and explicit iterative methods (including Euler's and Heun's method) used in temporal and spatial discretization for the approximate solutions of ordinary differential equations [3]. An important factor of Runge-Kutta methods is the order of the method. There has been a preferred development of higher order Runge-Kutta methods over the investigation of other versions within a specific order. One possible explanation of this is because, as Butcher [9] refers, higher order methods are able to provide, with a higher accuracy and relatively lower computational cost, the approximations of solutions to differential equations than lower order methods. Butcher [8,9,12] reported that explicit methods of orders one through ten have been developed. Tsitouras [13] constructed an explicit Runge-Kutta pair of orders 9 and 8. Ketcheson and Waheed [14] included a twelfth order method, (developed by Ono) in their study, and thereby confirm the existence of such order method. According to Hairer et al. [15], much research has been undertaken to choose the best versions from the infinite versions of fourth

order RK methods. An example of which is the popular, albeit obsolete, method of Gill (1951), which selected a set of RK coefficients that required fewer space in the computer memory. However, a review of literature for the existence of fourth order versions, revealed a relatively low number existed [1,3,12,16–20] of which, the classical fourth order RK method is the most used version.

First, despite the fact that the error term of all versions of a specific order of Runge-Kutta method have the same order ($O(h^p)$), the actual value of the error is dependent on the coefficients of the version of the method used. Hence, it is possible to select coefficients, in other words versions, such that the errors terms become as small as possible [15]. Next, the higher order methods are expected to be more accurate than the lower order methods, but their construction becomes more challenging as the order is increased. Consequently, the number of derivatives computed increases with the order of the version and this may result in unacceptable computational costs (time or resources wise). In fact, above fourth order RK methods, the number of derivatives computed is greater than the value of the order. For example, to obtain a fifth order accurate explicit RK method, a minimum of six evaluations of the derivatives (stages) are required. Finally, there exists inadequate literature on the detailed comparison of several versions of fourth order Runge-Kutta methods. In light of the previously highlighted points, this study, to investigate selected versions of fourth order explicit RK methods, was developed.

2. METHODOLOGY

2.1 Materials

Table 1 presents the crucial software and hardware components required for a successful completion of the study.

2.2 Harmonically Excited Duffing Oscillator

The Duffing equation, a second order, non-linear differential equation, has the general form:

$$\ddot{x} + \gamma\dot{x} + \alpha x + \beta x^3 = \mathrm{P_o}\sin(\omega t) \quad (1)$$

For this study, the parameters as defined in Table 2 were investigated. Inserting the values of parameters with single values into the general form given by equation (1) gives:

$$\ddot{x} + \gamma\dot{x} - \frac{x}{2}(1 - x^2) = \mathrm{P_o}\sin(\omega t) \quad (2)$$

Table 1. Materials utilized in the study

FORTRAN	Compiler Name	GNU FORTRAN Compiler [21]
	Version	**4.9.2**
Computer	Processor	Intel Pentium CPU B980 @ 2.40 Giga Hertz [Dual Core]
	System Type	64 bit Operating System, x64 – based processor

Table 2. Values of parameters and initial conditions of the Duffing equation

Parameters	Values	Description
γ	$0.0168 \leq \gamma \leq 0.168$	101 points, uniformly distributed between the intervals. Hence, an increment of 0.001512
α	$-\frac{1}{2}$	Unvaried
β	$+\frac{1}{2}$	Unvaried
P_o	$0.1 \leq P_o \leq 0.21$	101 points, uniformly distributed between the intervals. Hence, an increment of 0.0011
ω	$(0.2 \quad 1.0 \quad 2.0)$	Three excitation frequencies [22].
$(t_i, x_i, \dot{x}_i)$	$(0,-1,0),(0,0,0),(0,1,0)$	Equilibrium points of Equation (2), used as initial conditions

Table 3 lists some constants and expressions required for the simulation of the Duffing equation.

2.3 Versions of Explicit Fourth Order Runge-Kutta (RK) Method

The explicit fourth order RK methods are RK methods of order four ($O(h^4)$) and four stages (four computations of derivatives). They are the highest order achievable with the same number of stages as order, hence their popularity. In Table 4 is the set of equations defining the fourth order method as well as a Butcher tableau of the coefficients of the method.

Butcher [12] proved that the order conditions and simplifying assumptions for the fourth order RK methods result in eleven algebraic equations that can be used to solve for eleven out of the thirteen unknown coefficients in the fourth order Runge-Kutta formula (Table 4). The remaining two constants, which can be assigned any value, are responsible for the existence of an infinite number of versions of fourth order Runge-Kutta methods. Note that this phenomenon (infinite number of versions) is found in both lower and higher order methods. Efforts made by W. Kutta produced a complete classification of fourth order methods [9] and this classification, used as a basis for the selection of versions in this study, is presented in Table 5.

Assignment of a value to the appropriate free parameter (degree of freedom, DOF) results in a version of fourth order RK method. For this study, fifty values, (such that $0 < value < 1$), were randomly generated and assigned to the DOF, as outlined in Table 6. Five additional versions, obtained from literature [1, 12] were included, to make a total of fifty five versions. Noteworthy is RK_55, which is the Classical fourth order RK method.

Table 3. Other requirements for the simulation of the Duffing equation

Parameter	Value	Parameter	Value
Pi, π	3.141592654	Initial length of step, h_i	$\frac{T}{500}$
Excitation period, T	$2 * \frac{\Pi}{\omega}$	Unsteady period of Oscillation, uST	$50 * T$
Number of excitation periods, nT	500	Required accuracy, Eq	$1.0E-06$
Total simulation time, T_t	$T * nT$	Safety factor, S	0.95

Table 4. Definition of an explicit fourth order Runge-Kutta method

Equations	Butcher tableau				
$x_{i+1} = x_i + (\phi(x,t) \cdot h)$	0	0			
	c_2	a_{21}	0		
$\phi(x,t) = \sum_{i=1}^{s=4}(b_i \cdot k_i)$	c_3	a_{31}	a_{32}	0	
	c_4	a_{41}	a_{42}	a_{43}	0
$k_i = f\begin{pmatrix} t_i + (c_j \cdot h), \\ x_i + \sum_{m=1}^{s=4}(a_{jm} \cdot k_m \cdot h) \end{pmatrix}$		b_1	b_2	b_3	b_4

Table 5. Cases in which a solution is certain to exist as identified by Kutta

Case	c_2	c_3	Others	DOF
1	$c_2 \notin \left\{0, \frac{1}{2}, \frac{1}{2} \pm \frac{\sqrt{3}}{6}, 1\right\}$	$c_3 = 1 - c_2$	NA	c2
2	$c_2 \neq 0$	$c_3 = \frac{1}{2}$	$b_2 = 0$	c2
3	$c_2 = \frac{1}{2}$	$c_3 = 0$	$b_3 \neq 0$	b3
4	$c_2 = 1$	$c_3 = \frac{1}{2}$	$b_4 \neq 0$	b4
5	$c_2 = c_3 = \frac{1}{2}$	$c_3 = c_2 = \frac{1}{2}$	$b_3 \neq 0$	b3

2.4 Simulation

A simulation entails the extrapolation of $x_{i+1}(t_{i+1})$ from $x_i(t_i)$ over a step size, $h = t_{i+1} - t_i$ and two types of simulations are possible;

- Constant simulation: utilizes the same step size to increment the time throughout the simulation. The value, $h_{constant}$ obtained from [23] and given by equation (3) was used for the step size.

$$h_{constant} = \frac{\text{excitation period, } T}{500} \tag{3}$$

- Adaptive simulation: adjusts the step size based on the value of the estimated local truncation error, Er given by equation (4) and the accuracy required, Eq (see Table 3). As the aim is to compare two versions of the same order, the step-halving method was selected for the adaptive simulation. Then the value, $h_{constant}$ was used as the first step size, and for subsequent steps, the step size is adjusted using equation either (5) or (6). Equation (5) is used to decrease the step size (when $Er > Eq$) while equation (6) is used to increase the step size (when $Er \leq Eq$).

$$Er = |value\ from\ single\ step - value\ from\ step\ halving| \tag{4}$$

$$h_{new} = S * h * \left(\left(\frac{Eq}{Er}\right)^{(0.25)}\right) \tag{5}$$

$$h_{new} = S * h * \left(\left(\frac{Eq}{Er}\right)^{(0.20)}\right) \tag{6}$$

where S is the Safety factor, Er is the estimated error and h_{new} is the adjusted step size, which would be used for the next step. In this study, the failed steps, which are steps in which the estimated error is greater than the required accuracy $(Er > Eq)$, are rejected. That is, the values obtained during the step were discarded and the step repeated until the step was successful $(hence\ Er \leq Eq)$. Also in the successful steps, the values from the single step are used for the next step.

Defining a system as the Duffing equation, with all parameters assigned and one initial condition, then the varied parameters of the equation, along with the initial (equilibrium) conditions (see Table 2) culminate in a total of 91,809 unique systems to which a System Number (S_N) is assigned to (see Table 7). For example, the system defined by $S\_N = 1$ has $t_i = 0, x_i = 0, \dot{x}_i = 0, \omega = 1, \gamma = 0.0168, P_o = 0.1$ (see Table 2, Table 3, Table 7 and Table 8). Furthermore, these systems may be grouped into sets, depending on their combinations of initial conditions and excitation frequencies. This produced 9 sets, with 10,201 systems in each. Table 8 shows the sets, along with other details which are properties of every system in each set. Box A (in Table 8) shows the assignment of $'\gamma'$ and $'P_o'$ in a set.

The grouping into sets shown in Table 8 is the basis for the presentation of the results in a parameter plane.

2.5 FORTRAN Program

A FORTRAN program, utilizing double precision for the floating point (REAL) variables, was developed following the algorithm described in Fig. 1. The program was run for each system in Table 7 using the selected versions of the fourth order RK method. The results recorded include the number of successful steps, accepted step sizes, $[t,\ x,\ \dot{x}]$ data, number of failed (discarded) steps and the rejected step sizes at points $'R1', 'R1', 'R2', 'R3', 'R3'$ respectively in Fig. 1. These results are presented next.

Other useful information related to the simulation can be accessed on Github [27].

3. RESULTS AND DISCUSSION

3.1 Program Validation

Validation of the FORTRAN program was achieved by comparing the Poincaré section plotted with data from the program against Poincaré section found in literature [24, 25] and it was found to be identical, thereby validating the program. Fig. 2 (a) was obtained with a constant simulation run for $S\_N = 10201$, (hence $t_i = 0, x_i = 0, \dot{x}_i = 0, \omega = 1.0, \gamma = 0.168, P_o = 0.21$) and $nT = 2500$.

3.2 Safety Factor and Step Size Selection

Fig. 2 (b) presents the number of step ratio for a range of Safety factor values, from $S = 0.2$ to $S = 1$ for selected RK versions (RK_1,

RK_2, RK_8, RK_14, RK_51, RK_52, RK_53, RK_54 and RK_55). On the vertical axis is the number of step ratio defined as the ratio of the number of successive steps achieved with an adaptive simulation to that achieved with a constant simulation, expressed in percentage.

$$number\ of\ step\ ratio = \frac{number\ of\ successive\ steps\ with\ adaptive\ simulation}{number\ of\ successive\ steps\ with\ constant\ simulation}\ \% \quad (7)$$

Table 6. Specification of free parameters for Runge-Kutta versions (RK)

Case	1		2		3		4		5	
(DOF)	C2		C2		B3		B4		B3	
Randomly generated values	RK_1	0.7513	RK_11	0.8407	RK_21	0.0759	RK_31	0.1622	RK_41	0.4505
	RK_2	0.2551	RK_12	0.2543	RK_22	0.0540	RK_32	0.7943	RK_42	0.0838
	RK_3	0.5060	RK_13	0.8143	RK_23	0.5308	RK_33	0.3112	RK_43	0.2290
	RK_4	0.6991	RK_14	0.2435	RK_24	0.7792	RK_34	0.5285	RK_44	0.9133
	RK_5	0.8909	RK_15	0.9293	RK_25	0.9340	RK_35	0.1656	RK_45	0.1524
	RK_6	0.9593	RK_16	0.3500	RK_26	0.1299	RK_36	0.6020	RK_46	0.8258
	RK_7	0.5472	RK_17	0.1966	RK_27	0.5688	RK_37	0.2630	RK_47	0.5383
	RK_8	0.1386	RK_18	0.2511	RK_28	0.4694	RK_38	0.6541	RK_48	0.9961
	RK_9	0.1493	RK_19	0.6160	RK_29	0.0119	RK_39	0.6892	RK_49	0.0782
	RK_10	0.2575	RK_20	0.4733	RK_30	0.3371	RK_40	0.7482	RK_50	0.4427
Literature	RK_51	0.3333	RK_52	0.2500	RK_53	0.0833	RK_54	0.1667	RK_55	0.3333

Table 7. Format of data for (independent) and from (dependent) simulation

S_N	Independent data				Dependent data				
	ω	γ	$(x_i, \dot{x}_i)$	P_o	RK_1	RK_2	⋯	RK_54	RK_55
1	Assigned parameters				Results recorded from investigation				
2	and initial conditions				(e.g. Number of successive steps				
⋮	(see Table 2)				to simulation end)				
91809									
Unique values	3	101	3	101					

Table 8. Summary of Initial conditions and parameters for each system simulated

S_N	Set	t_i	x_i	$\dot{x}_i$	ω	T	T_t	$h_{constant}$
1 – 10201	Set 1	0	0	0	1.0	6.2832	3141.5900	1.2566E-02
10202 – 20402	Set 2	0	1	0	1.0	6.2832	3141.5900	1.2566E-02
20403 – 30603	Set 3	0	-1	0	1.0	6.2832	3141.5900	1.2566E-02
30604 – 40804	Set 4	0	0	0	2.0	3.1416	1570.8000	6.2832E-03
40805 – 51005	Set 5	0	1	0	2.0	3.1416	1570.8000	6.2832E-03
51006 – 61206	Set 6	0	-1	0	2.0	3.1416	1570.8000	6.2832E-03
61207 – 71407	Set 7	0	0	0	0.2	31.4159	15708.0000	6.2832E-02
71408 – 81608	Set 8	0	1	0	0.2	31.4159	15708.0000	6.2832E-02
81609 – 91809	Set 9	0	-1	0	0.2	31.4159	15708.0000	6.2832E-02

Box A					
	Set 1	$\gamma = 0.016800$	$\gamma = 0.018312$	⋯	$\gamma = 0.168000$
	$P_o = 0.2100$	**S_N** = 10101	**S_N** = 10102	⋯	**S_N** = 10201
	⋮	⋮	⋮	⋱	⋮
	$P_o = 0.1011$	**S_N** = 102	**S_N** = 103	⋯	**S_N** = 202
	$P_o = 0.1000$	**S_N** = 1	**S_N** = 2	⋯	**S_N** = 101

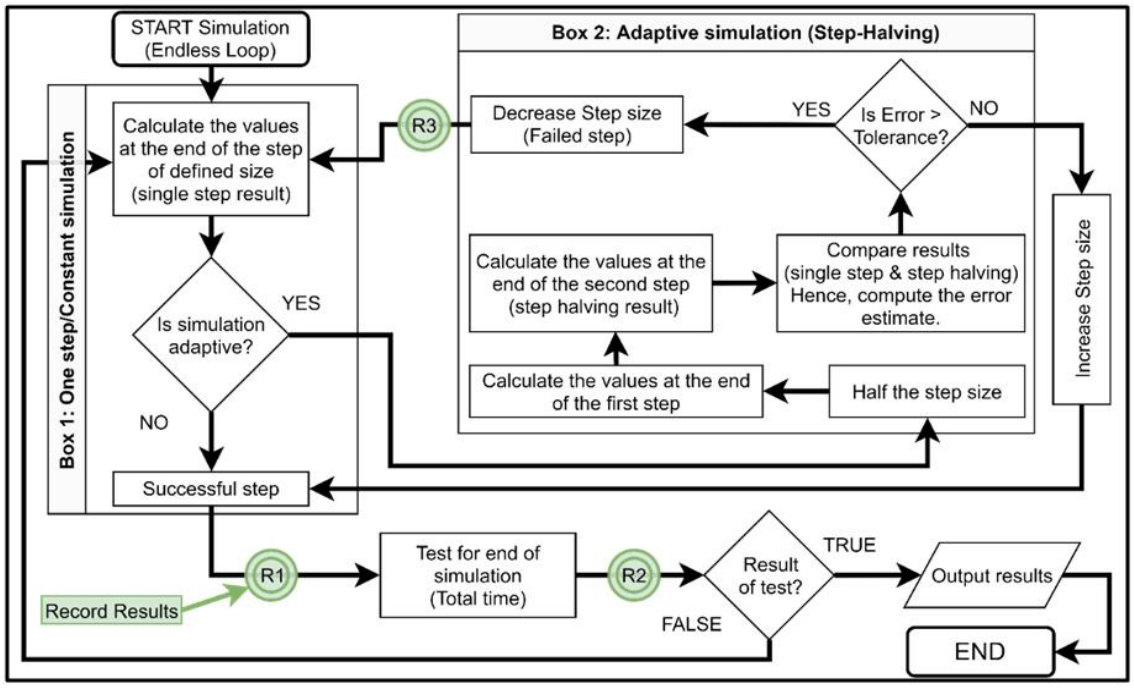

Fig. 1. Algorithm for the simulation of the Duffing equation

A value of $S = 1$ is susceptible to constant rejection and isn't advised [26], while values of $S > 1$ results in an infinite loop, as h_{new} is never accepted. Hence, considering Fig. 2, a value of $0.5 < S < 1$ is advised. Note that this study utilized $S = 0.95$ (see Table 3).

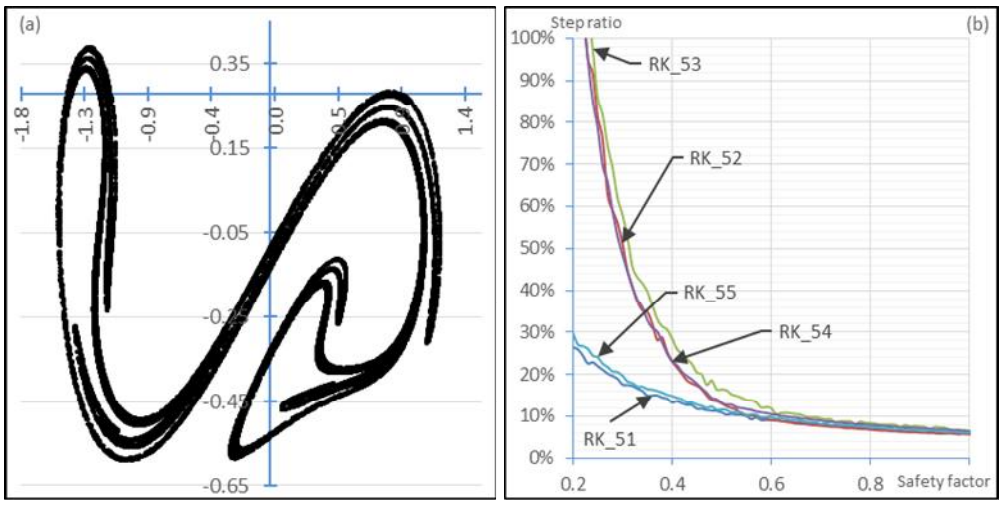

Fig. 2. (a) Poincaré section obtained with $t_i = 0, x_i = 0, \dot{x}_i = 0, \omega = 1.0, \gamma = 0.168, P_o = 0.21, nT = 25\ 00$ (b) Variation of number of successive steps to simulation end with safety factor

3.3 Runge-Kutta Versions

Two RK methods of the same order differ in their coefficients (see Table 4). Fig. 3 illustrates graphically the consequence of the differences between two versions of fourth order RK methods with regards to the time, 't', step size, 'h' and number of failed (discarded) steps, '$NofFS$' within the first forty successful steps of the adaptive simulation execution. The period of excitation, '$T = 6.2832s$' is also shown, so as to ascertain the point at which the time is reached. From Fig. 3(a), it can be observed that RK_1 required 37 steps for $t = 6.2832s$ and had a total of 54 steps, since it rejected 14 steps. On the other hand in Fig. 3(b), it can be observed that RK_55 (classical 4th RK) needed 30 steps for $t = 6.2832s$, had a total of 49 steps, rejecting only 9 steps. The step sizes, h provide an explanation for the difference in number of successive steps to $t = T$, as it can be observed that RK_55 permitted higher values of h than RK_1.

3.4 Parameter Plane

The parameter plane is a plot of two of the investigated parameters of systems in a given set. Each point on the parameter plane corresponds to a value of the forcing amplitude and the damping constant. As 101 values each, of the forcing amplitude and damping constant were selected, a total of 10,201 points exist on the plane. The colour (as well as shape) of a point indicates a Runge-Kutta version, with the assignment of colours (shapes) dependent on the results from the simulation. Fig. 4, Fig. 5, Fig. 6, Fig. 7 and Fig. 8 are parameter planes showing the high performing versions for a given set, where a version is considered to be high performing if it achieves the lowest number of successful steps between the start and end of adaptive simulation periods for a given system. Table 9 gives numerical information on the respective parameter planes for each Set. The frequency of a version (Freq) is the number of systems in which the version had the lowest number of steps, the percentage share (% share) is the fraction of the total investigated systems (expressed in percentage) in which a version had the lowest number of steps and for the Case of the version, see Table 5.

Recall the definition of a Set (see Table 8), then, the parameter planes in Fig. 4(a), Fig. 4(b), and Fig. 5(a) have the same excitation frequency, $\omega = 1.0$. Similarly, Fig. 5(b), Fig. 6(a) and Fig. 6(b), $\omega = 2.0$ while Fig. 7(a), Fig. 7(b) and Fig. 8, $\omega = 0.2$. It is observable that within parameter planes with the same excitation frequency, the version (colour/shape) distribution is identical and this implies that the version performance is independent of initial conditions. The data in the Table 9 confirms this, where we find that the same set of versions have the largest percentage share for each excitation frequency, regardless of the initial conditions. On the other hand, across the excitation periods, the version

distributions are not identical, instead, the distribution becomes less coarse as the excitation frequency is increased ($\omega = \{0.2 \rightarrow 1.0 \rightarrow 2.0\}$). Also, the total percentage share of the versions increases as well. This confirms the dependence of version performance on excitation frequency. As for the versions, no version had high performance in all three excitation frequencies but across two frequencies , $\omega = 0.2$ and $\omega = 1.0$, some high performing versions overlapped. With regards to the Cases (Table 5), majority of the high performing versions were selected from Cases 1 and 2.

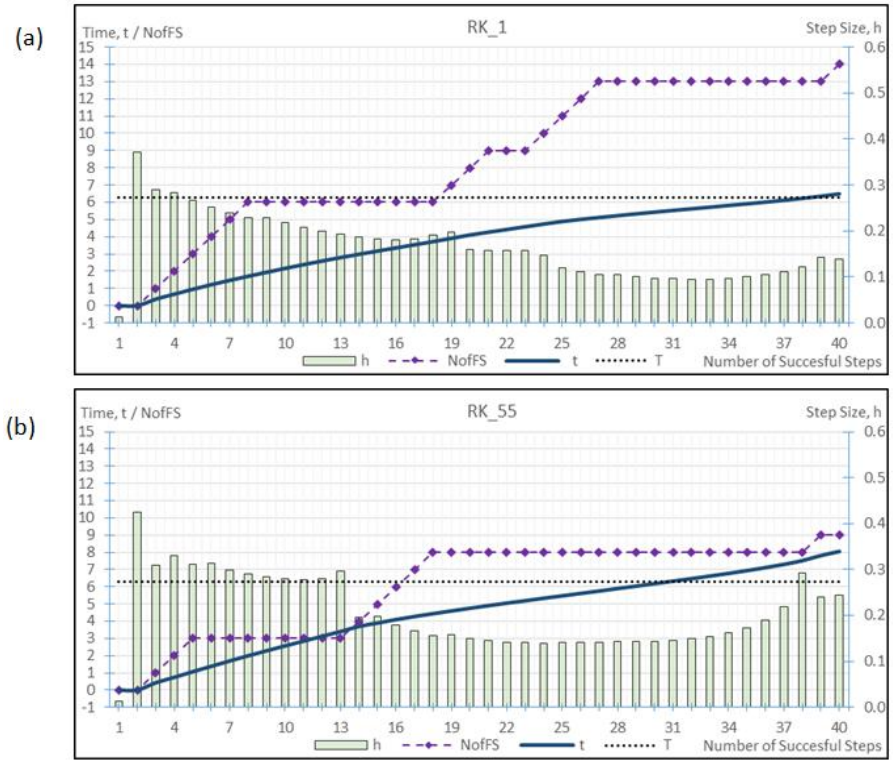

Fig. 3. Time series for a period of oscillation for the Duffing equation for (a) RK_1 (b) RK_55. Each shows the adaptive step size and number of failed steps (NofFS) for each step.

Table 9. Selected high performing versions in all Sets (hence 1 to 9)

Set 01			Set 02			Set 03		
RK_V	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**
RK_17	6884	67.48%	RK_17	6150	60.29%	RK_17	6160	60.39%
RK_2	2084	20.43%	RK_2	2399	23.52%	RK_2	2481	24.32%
RK_10	342	3.35%	RK_10	441	4.32%	RK_10	392	3.84%
RK_14	239	2.34%	RK_14	349	3.42%	RK_14	342	3.35%
RK_12	190	1.86%	RK_52	244	2.39%	RK_12	228	2.24%
Total	9739	95.46%	Total	9583	93.94%	Total	9603	94.14%
Set 04			**Set 05**			**Set 06**		
RK_V	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**
RK_20	5801	56.87%	RK_20	5921	58.04%	RK_20	6299	61.75%
RK_16	3344	32.78%	RK_16	3327	32.61%	RK_16	3058	29.98%
RK_55	633	6.21%	RK_3	442	4.33%	RK_55	600	5.88%
RK_3	352	3.45%	RK_55	425	4.17%	RK_3	148	1.45%
RK_50	33	0.32%	RK_50	54	0.53%	RK_50	60	0.59%
Total	10163	99.63%	Total	10169	99.68%	Total	10165	99.65%

Set 07			Set 08			Set 09		
RK_V	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**	**RK_V**	**Freq**	**% Share**
RK_8	4401	43.14%	RK_8	5851	57.36%	RK_8	6137	60.16%
RK_17	3103	30.42%	RK_17	1608	15.76%	RK_17	1612	15.80%
RK_2	666	6.53%	RK_16	647	6.34%	RK_16	579	5.68%
RK_14	428	4.20%	RK_12	628	6.16%	RK_12	534	5.23%
RK_12	325	3.19%	RK_2	522	5.12%	RK_2	439	4.30%
Total	8923	87.48%	Total	9256	90.74%	Total	9301	91.17%

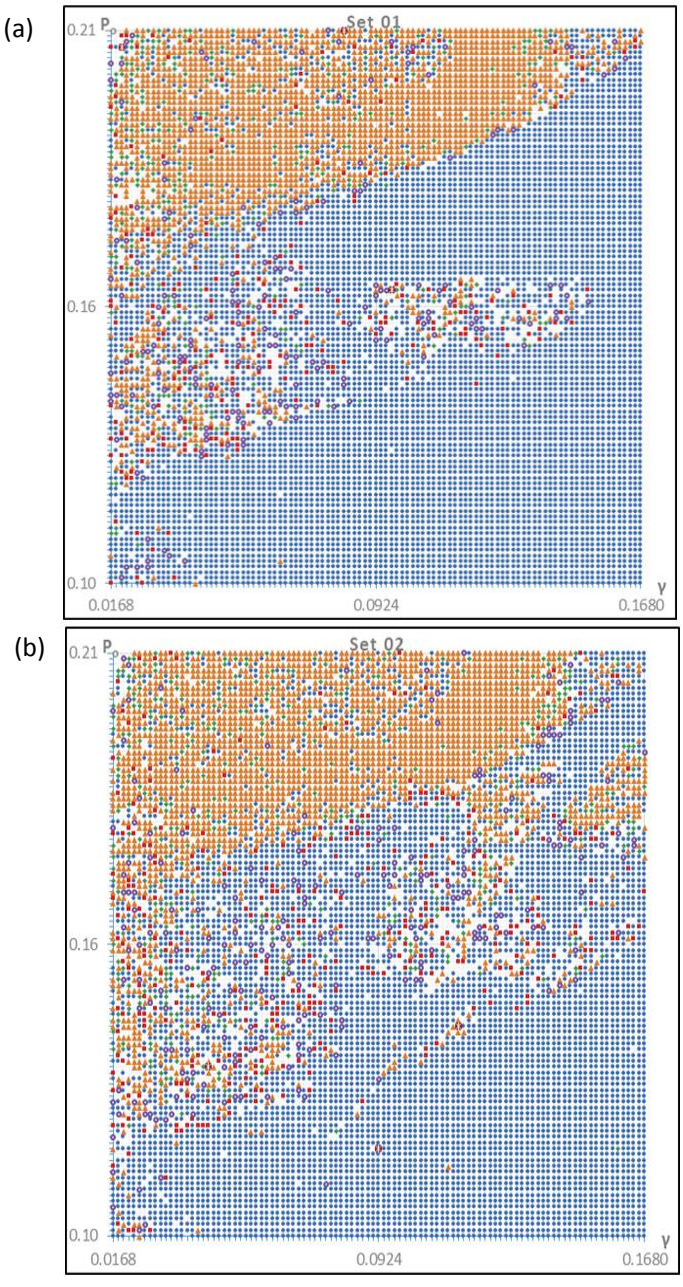

Fig. 4. Parameter plane highlighting the (five) versions with the lowest number of steps when used to simulate the dynamics of systems in (a) Set 1 (b) Set 2

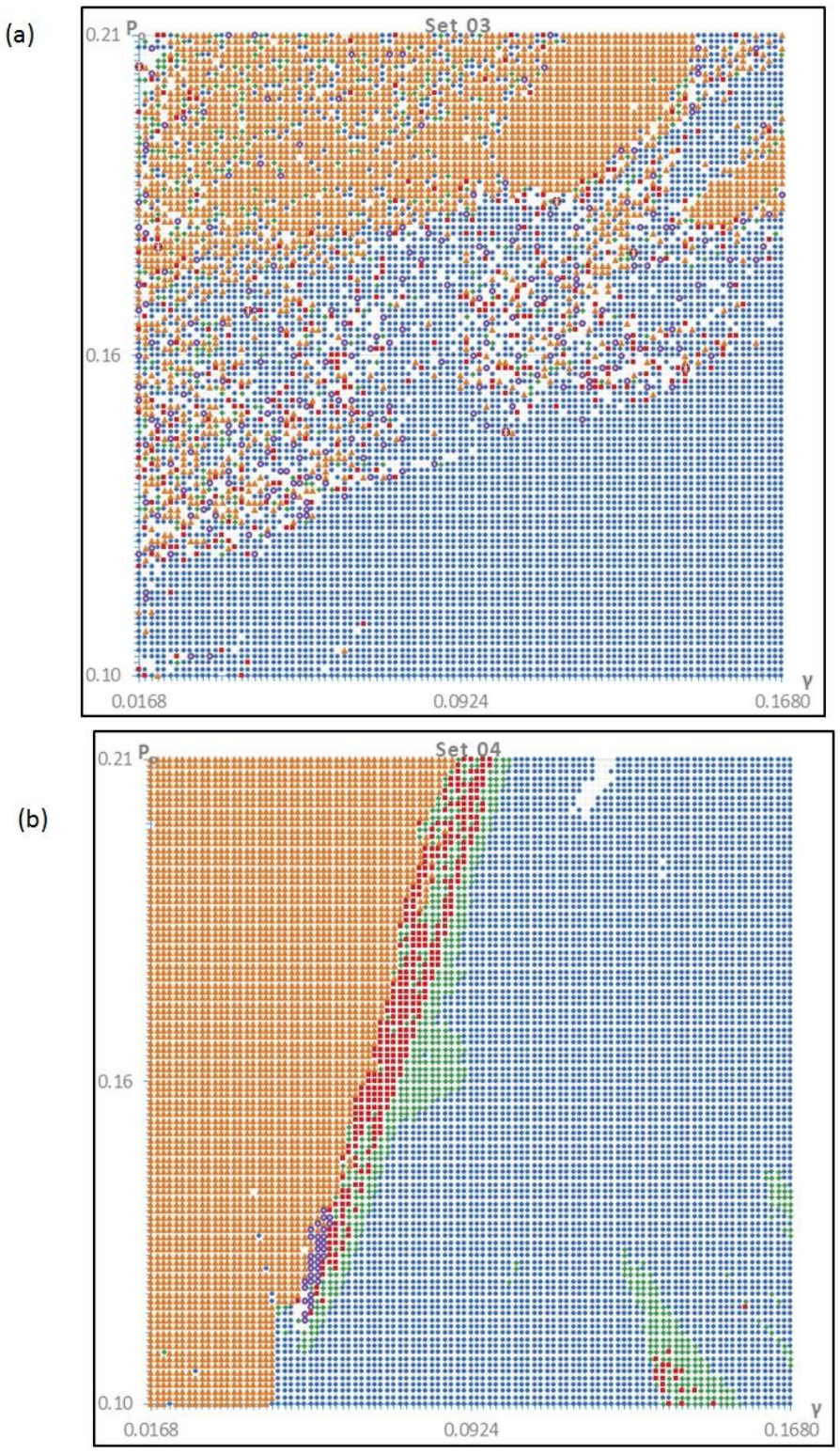

Fig. 5. Parameter plane highlighting the (five) versions with the lowest number of steps when used to simulate the dynamics of systems in (a) Set 3 (b) Set 4

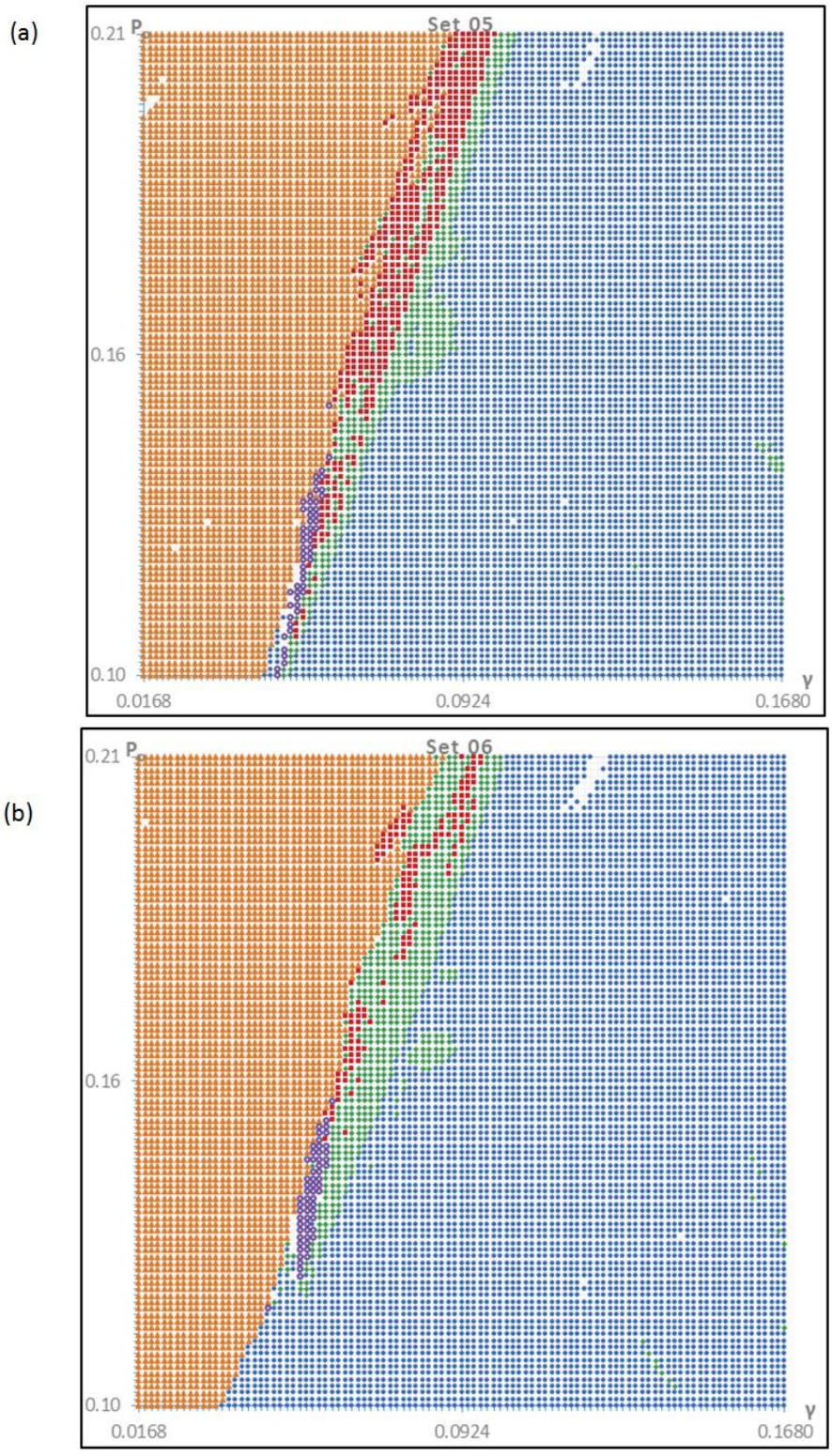

Fig. 6. Parameter plane highlighting the (five) versions with the lowest number of steps when used to simulate the dynamics of systems in (a) Set 5 (b) Set

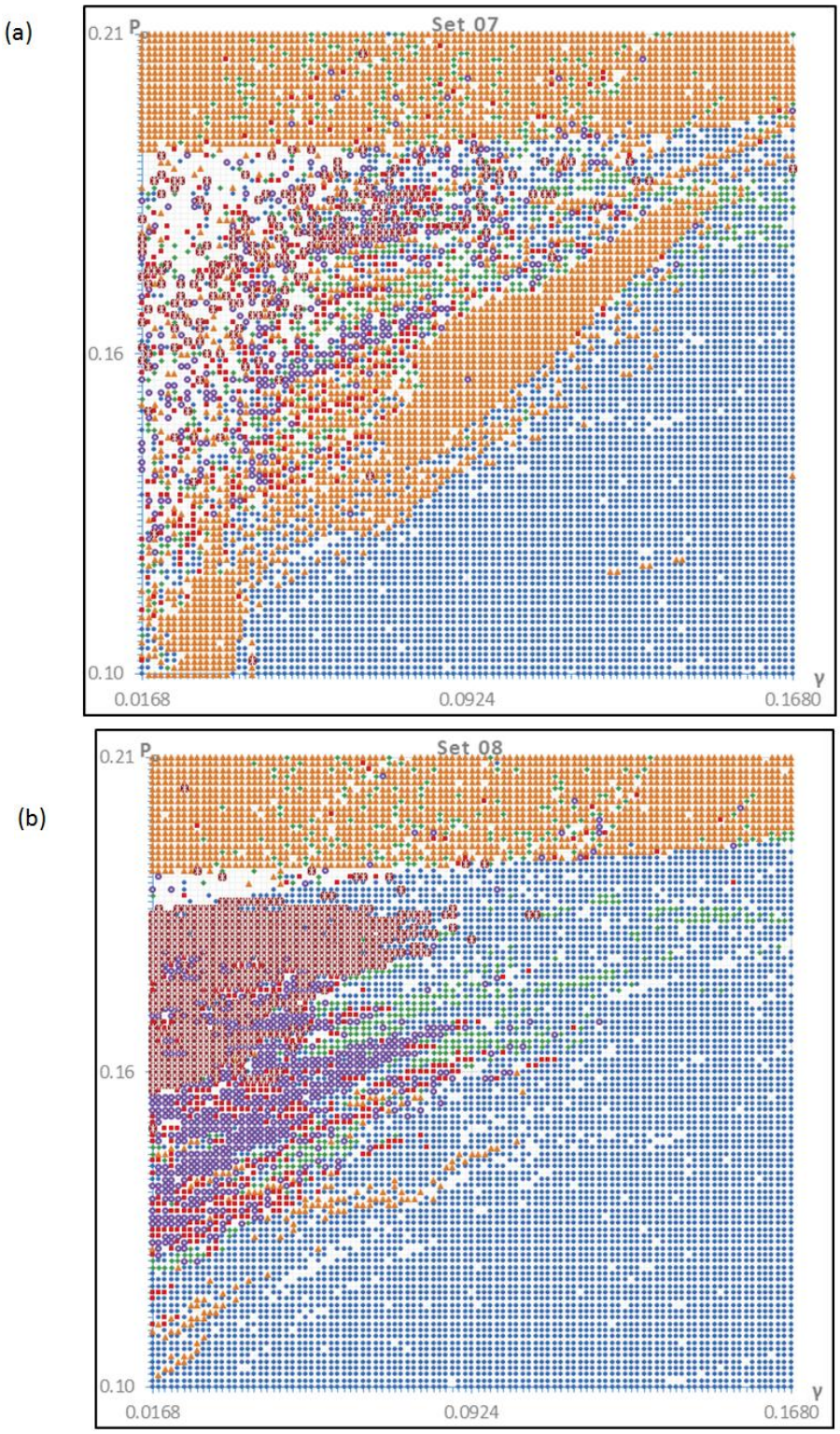

Fig. 7. Parameter plane highlighting the (five) versions with the lowest number of steps when used to simulate the dynamics of systems in (a) Set 7 (b) Set 8

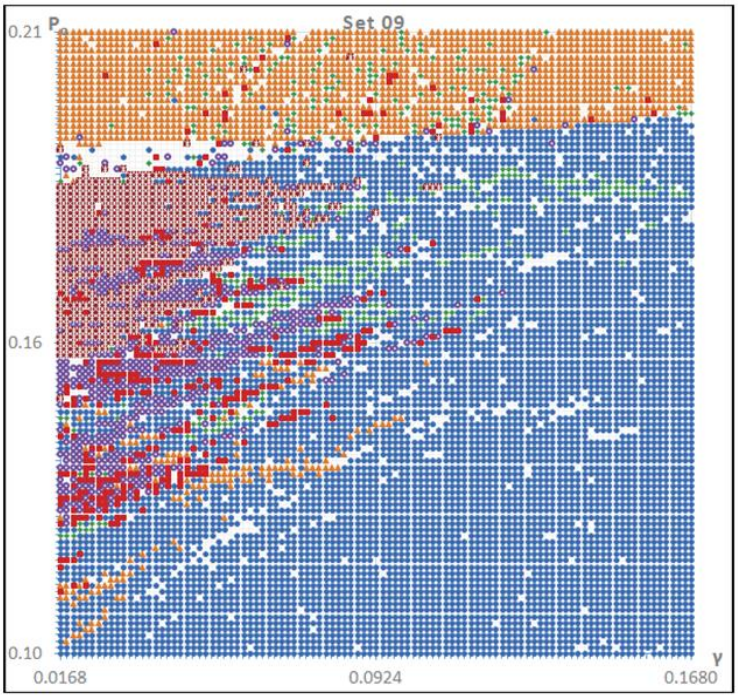

Fig. 8. Parameter plane highlighting the (five) versions with the lowest number of steps when used to simulate the dynamics of systems in Set 9

Sets	Legend (Fig. 3 – Fig. 8)
1, 2, 3	RK_17, RK_2, RK_10, RK_14, RK_12, RK_55
4, 5, 6	RK_20, RK_16, RK_55, RK_3, RK_50, RK_52
7, 8, 9	RK_8, RK_17, RK_2, RK_14, RK_12, RK_16

4. CONCLUSION

Investigation of different versions of fourth order Runge-Kutta methods as simulation tools for seeking the solution of harmonically excited Duffing's oscillator yielded results from which one can conclude that there exists significant variations when versions of the same order are used for the solution of ordinary differential equations. Other inferences from the results include:

- The number of successive steps between start and end of simulation periods vary with versions
- The version performance, with regards to the number of successive steps, is highly dependent on the excitation frequency, as the same set of versions have high performance across systems with the same excitation frequency.
- The study results indicated strong preference for Runge-Kutta version derivation among two derivation options studied.
- The popular classical fourth order Runge-Kutta version is not the fastest, as other versions from the selection made for this study, outperformed it.

COMPETING INTERESTS

Authors have declared that no competing interests exist

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