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Closed Formulas for the Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^{n} W_k^3$ and $\sum_{k=1}^{n} W_{-k}^3$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas for the cubes of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas.

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1 INTRODUCTION

The Fibonacci and Lucas sequences are very well-known examples of second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci and Lucas sequences are a source of many nice and interesting identities. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2, \quad F_0 = 0, \; F_1 = 1.$$

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and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \ \ n \geq 2, \quad L_0 = 2, \ L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \ \ W_0 = a, \ W_1 = b, \ \ (n \geq 2) \quad (1.1)$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	Notation: $W_n(a, b; r, s)$	OEIS: [5]
Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551

The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n} F_k^3 = \frac{1}{2}(-F_{n+2}^3 - 3F_{n+1}^3 + 3F_{n+2}^2F_{n+1} + 1)$$

and

$$\sum_{k=0}^{n} P_k^3 = \frac{1}{14}(-2P_{n+2}^3 - 16P_{n+1}^3 + 9P_{n+2}^2P_{n+1} - 3P_{n+1}^2P_{n+2} + 2).$$

In this work, we derive expressions for sums of third powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[6,7,8,9,10,11]	[12,13]	[14,15,16]
Generalized Tribonacci	[17]		
Generalized Tetranacci	[18,19]		

2 SUM FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$ we have the following formulas: If $(r + s - 1)(rs - s^3 + 1)(r + s - rs + r^2 + s^2 + 1) \neq 0$ then*

(a)

$$\sum_{k=0}^{n} W_k^3 = \frac{\Delta_1}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_1 \quad = \quad & -(s^3+2rs-1)W_{n+2}^3 - (r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_{n+1}^3 \\ & +3rs(r+s^2)W_{n+2}^2W_{n+1} - 3rs^2\,(rs-1)\,W_{n+1}^2W_{n+2} + (2rs+s^3-1)W_1^3 \\ & +(r^4s+3r^2s^2-r^3s^3+2rs+r^3+s^3-1)W_0^3 - 3rs(r+s^2)W_1^2W_0 + 3rs^2(rs-1)W_0^2W_1.\end{aligned}$$

(b)

$$\sum_{k=0}^{n} W_k^2W_{k+1} = \frac{\Delta_2}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_2 \quad = \quad & -r\,(rs-1)\,W_{n+2}^3 - rs^3\,(rs-1)\,W_{n+1}^3 + s(2r^3-s^3+1)W_{n+2}^2W_{n+1} \\ & -(-2rs^4+r^4s+2rs+r^3+s^3-1)W_{n+1}^2W_{n+2} + r\,(rs-1)\,W_1^3 \\ & +rs^3\,(rs-1)\,W_0^3 - s(2r^3-s^3+1)W_1^2W_0 + (-2rs^4+r^4s+2rs+r^3+s^3-1)W_0^2W_1.\end{aligned}$$

(c)

$$\sum_{k=0}^{n} W_{k+1}^2W_k = \frac{\Delta_3}{(r+s-1)(rs-s^3+1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_3 \quad = \quad & r(r+s^2)W_{n+2}^3 + rs^3(r+s^2)W_{n+1}^3 - (3r^2s^2+r^3+s^3-1)W_{n+2}^2W_{n+1} \\ & +s^2(2r^3-s^3+1)W_{n+1}^2W_{n+2} - r(r+s^2)W_1^3 - rs^3(r+s^2)W_0^3 \\ & +(3r^2s^2+r^3+s^3-1)W_1^2W_0 - s^2(2r^3-s^3+1)W_0^2W_1.\end{aligned}$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$s^3W_n^3 = W_{n+2}^3 - 3rW_{n+2}^2W_{n+1} + 3r^2W_{n+1}^2W_{n+2} - r^3W_{n+1}^3$$

and so

$$\begin{aligned} s^3W_n^3 &= W_{n+2}^3 - 3rW_{n+2}^2W_{n+1} + 3r^2W_{n+1}^2W_{n+2} - r^3W_{n+1}^3 \\ s^3W_{n-1}^3 &= W_{n+1}^3 - 3rW_{n+1}^2W_n + 3r^2W_n^2W_{n+1} - r^3W_n^3 \\ s^3W_{n-2}^3 &= W_n^3 - 3rW_n^2W_{n-1} + 3r^2W_{n-1}^2W_n - r^3W_{n-1}^3 \\ & \vdots \\ s^3W_2^3 &= W_4^3 - 3rW_4^2W_3 + 3r^2W_3^2W_4 - r^3W_3^3 \\ s^3W_1^3 &= W_3^3 - 3rW_3^2W_2 + 3r^2W_2^2W_3 - r^3W_2^3 \\ s^3W_0^3 &= W_2^3 - 3rW_2^2W_1 + 3r^2W_1^2W_2 - r^3W_1^3 \end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned} s^3\sum_{k=0}^{n} W_k^3 \quad = \quad & (W_{n+2}^3 + W_{n+1}^3 - W_1^3 - W_0^3 + \sum_{k=0}^{n} W_k^3) - 3r(W_{n+2}^2W_{n+1} - W_1^2W_0 + \sum_{k=0}^{n} W_{k+1}^2W_k) \\ & +3r^2(W_{n+1}^2W_{n+2} - W_0^2W_1 + \sum_{k=0}^{n} W_k^2W_{k+1}) - r^3(W_{n+1}^3 - W_0^3 + \sum_{k=0}^{n} W_k^3). \end{aligned} \tag{2.1}$$

Next we calculate $\sum_{k=0}^{n} W_{k+1}^2 W_k$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$sW_{n+1}^2 W_n = W_{n+1}^2 W_{n+2} - rW_{n+1}^3$$

and so

$$\begin{aligned}
sW_{n+1}^2 W_n &= W_{n+1}^2 W_{n+2} - rW_{n+1}^3 \\
sW_n^2 W_{n-1} &= W_n^2 W_{n+1} - rW_n^3 \\
sW_{n-1}^2 W_{n-2} &= W_{n-1}^2 W_n - rW_{n-1}^3 \\
&\vdots \\
sW_3^2 W_2 &= W_3^2 W_4 - rW_3^3 \\
sW_2^2 W_1 &= W_2^2 W_3 - rW_2^3 \\
sW_1^2 W_0 &= W_1^2 W_2 - rW_1^3
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=0}^{n} W_{k+1}^2 W_k = (W_{n+1}^2 W_{n+2} - W_0^2 W_1 + \sum_{k=0}^{n} W_k^2 W_{k+1}) - r(W_{n+1}^3 - W_0^3 + \sum_{k=0}^{n} W_k^3). \tag{2.2}$$

Next we calculate $\sum_{k=0}^{n} W_k^2 W_{k+1}$. Again, using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1} \Rightarrow s^2 W_n^2 = W_{n+2}^2 + r^2 W_{n+1}^2 - 2rW_{n+2} W_{n+1}$$

we obtain

$$s^2 W_n^2 W_{n+1} = W_{n+2}^2 W_{n+1} + r^2 W_{n+1}^3 - 2rW_{n+1}^2 W_{n+2}$$

and so

$$\begin{aligned}
s^2 W_n^2 W_{n+1} &= W_{n+2}^2 W_{n+1} + r^2 W_{n+1}^3 - 2rW_{n+1}^2 W_{n+2} \\
s^2 W_{n-1}^2 W_n &= W_{n+1}^2 W_n + r^2 W_n^3 - 2rW_n^2 W_{n+1} \\
s^2 W_{n-2}^2 W_{n-1} &= W_n^2 W_{n-1} + r^2 W_{n-1}^3 - 2rW_{n-1}^2 W_n \\
&\vdots \\
s^2 W_2^2 W_3 &= W_4^2 W_3 + r^2 W_3^3 - 2rW_3^2 W_4 \\
s^2 W_1^2 W_2 &= W_3^2 W_2 + r^2 W_2^3 - 2rW_2^2 W_3 \\
s^2 W_0^2 W_1 &= W_2^2 W_1 + r^2 W_1^3 - 2rW_1^2 W_2
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
s^2 \sum_{k=0}^{n} W_k^2 W_{k+1} &= (W_{n+2}^2 W_{n+1} - W_1^2 W_0 + \sum_{k=0}^{n} W_{k+1}^2 W_k) + r^2(W_{n+1}^3 - W_0^3 + \sum_{k=0}^{n} W_k^3) \quad (2.3) \\
&\quad -2r(W_{n+1}^2 W_{n+2} - W_0^2 W_1 + \sum_{k=0}^{n} W_k^2 W_{k+1}).
\end{aligned}$$

Then, solving the system (2.1)-(2.2)-(2.3), the required results of (a),(b) and (c) follow.

Taking $r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.1. *If $r = s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^3 = \frac{1}{2}(-W_{n+2}^3 - 3W_{n+1}^3 + 3W_{n+2}^2 W_{n+1} + W_1^3 + 3W_0^3 - 3W_1^2 W_0)$.

(b) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{2}(W_{n+2}^2 W_{n+1} - W_{n+1}^2 W_{n+2} - W_1^2 W_0 + W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} W_{k+1}^2 W_k = \frac{1}{2}(W_{n+2}^3 + W_{n+1}^3 - 2W_{n+2}^2 W_{n+1} + W_{n+1}^2 W_{n+2} - W_1^3 - W_0^3 + 2W_1^2 W_0 - W_0^2 W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.2. *For $n \geq 0$, Fibonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} F_k^3 = \frac{1}{2}(-F_{n+2}^3 - 3F_{n+1}^3 + 3F_{n+2}^2 F_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} F_k^2 F_{k+1} = \frac{1}{2}(F_{n+2}^2 F_{n+1} - F_{n+1}^2 F_{n+2})$.

(c) $\sum_{k=0}^{n} F_{k+1}^2 F_k = \frac{1}{2}(F_{n+2}^3 + F_{n+1}^3 - 2F_{n+2}^2 F_{n+1} + F_{n+1}^2 F_{n+2} - 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.3. *For $n \geq 0$, Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} L_k^3 = \frac{1}{2}(-L_{n+2}^3 - 3L_{n+1}^3 + 3L_{n+2}^2 L_{n+1} + 19)$.

(b) $\sum_{k=0}^{n} L_k^2 L_{k+1} = \frac{1}{2}(L_{n+2}^2 L_{n+1} - L_{n+1}^2 L_{n+2} + 2)$.

(c) $\sum_{k=0}^{n} L_{k+1}^2 L_k = \frac{1}{2}(L_{n+2}^3 + L_{n+1}^3 - 2L_{n+2}^2 L_{n+1} + L_{n+1}^2 L_{n+2} - 9)$.

Taking $r = 2, s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.2. *If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^3 = \frac{1}{14}(-2W_{n+2}^3 - 16W_{n+1}^3 + 9W_{n+2}^2 W_{n+1} - 3W_{n+1}^2 W_{n+2} + 2W_1^3 + 16W_0^3 - 9W_1^2 W_0 + 3W_0^2 W_1)$.

(b) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{14}(-W_{n+2}^3 - W_{n+1}^3 + 8W_{n+2}^2 W_{n+1} - 12W_{n+1}^2 W_{n+2} + W_1^3 + W_0^3 - 8W_1^2 W_0 + 12W_0^2 W_1)$.

(c) $\sum_{k=0}^{n} W_{k+1}^2 W_k = \frac{1}{14}(3W_{n+2}^3 - 10W_{n+2}^2 W_{n+1} + 3W_{n+1}^3 + 8W_{n+1}^2 W_{n+2} - 3W_1^3 - 3W_0^3 + 10W_1^2 W_0 - 8W_0^2 W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.4. *For $n \geq 0$, Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k^3 = \frac{1}{14}(-2P_{n+2}^3 - 16P_{n+1}^3 + 9P_{n+2}^2 P_{n+1} - 3P_{n+1}^2 P_{n+2} + 2)$.

(b) $\sum_{k=0}^{n} P_k^2 P_{k+1} = \frac{1}{14}(-P_{n+2}^3 - P_{n+1}^3 + 8P_{n+2}^2 P_{n+1} - 12P_{n+1}^2 P_{n+2} + 1)$.

(c) $\sum_{k=0}^{n} P_{k+1}^2 P_k = \frac{1}{14}(3P_{n+2}^3 - 10P_{n+2}^2 P_{n+1} + 3P_{n+1}^3 + 8P_{n+1}^2 P_{n+2} - 3)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k^3 = \frac{1}{14}(-2Q_{n+2}^3 - 16Q_{n+1}^3 + 9Q_{n+2}^2 Q_{n+1} - 3Q_{n+1}^2 Q_{n+2} + 96)$.

(b) $\sum_{k=0}^{n} Q_k^2 Q_{k+1} = \frac{1}{14}(-Q_{n+2}^3 - Q_{n+1}^3 + 8Q_{n+2}^2 Q_{n+1} - 12Q_{n+1}^2 Q_{n+2} + 48)$.

(c) $\sum_{k=0}^{n} Q_{k+1}^2 Q_k = \frac{1}{14}(3Q_{n+2}^3 - 10Q_{n+2}^2 Q_{n+1} + 3Q_{n+1}^3 + 8Q_{n+1}^2 Q_{n+2} - 32).$

Taking $r = 1, s = 2$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.3. *If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k^3 = \frac{1}{70}(11W_{n+2}^3 + 18W_{n+1}^3 - 30W_{n+2}^2 W_{n+1} + 12W_{n+1}^2 W_{n+2} - 11W_1^3 - 18W_0^3 + 30W_1^2 W_0 - 12W_0^2 W_1).$

(b) $\sum_{k=0}^{n} W_k^2 W_{k+1} = \frac{1}{70}(W_{n+2}^3 + 8W_{n+1}^3 + 10W_{n+2}^2 W_{n+1} - 18W_{n+1}^2 W_{n+2} - W_1^3 - 8W_0^3 - 10W_1^2 W_0 + 18W_0^2 W_1).$

(c) $\sum_{k=0}^{n} W_{k+1}^2 W_k = \frac{1}{70}(-5W_{n+2}^3 - 40W_{n+1}^3 + 20W_{n+2}^2 W_{n+1} + 20W_{n+1}^2 W_{n+2} + 5W_1^3 + 40W_0^3 - 20W_1^2 W_0 - 20W_0^2 W_1).$

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.6. *For $n \geq 0$, Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k^3 = \frac{1}{70}(11J_{n+2}^3 + 18J_{n+1}^3 - 30J_{n+2}^2 J_{n+1} + 12J_{n+1}^2 J_{n+2} - 11).$

(b) $\sum_{k=0}^{n} J_k^2 J_{k+1} = \frac{1}{70}(J_{n+2}^3 + 8J_{n+1}^3 + 10J_{n+2}^2 J_{n+1} - 18J_{n+1}^2 J_{n+2} - 1).$

(c) $\sum_{k=0}^{n} J_{k+1}^2 J_k = \frac{1}{70}(-5J_{n+2}^3 - 40J_{n+1}^3 + 20J_{n+2}^2 J_{n+1} + 20J_{n+1}^2 J_{n+2} + 5).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.7. *For $n \geq 0$, Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} j_k^3 = \frac{1}{70}(11j_{n+2}^3 + 18j_{n+1}^3 - 30j_{n+2}^2 j_{n+1} + 12j_{n+1}^2 j_{n+2} - 143).$

(b) $\sum_{k=0}^{n} j_k^2 j_{k+1} = \frac{1}{70}(j_{n+2}^3 + 8j_{n+1}^3 + 10j_{n+2}^2 j_{n+1} - 18j_{n+1}^2 j_{n+2} - 13).$

(c) $\sum_{k=0}^{n} j_{k+1}^2 j_k = \frac{1}{70}(-5j_{n+2}^3 - 40j_{n+1}^3 + 20j_{n+2}^2 j_{n+1} + 20j_{n+1}^2 j_{n+2} + 205).$

3 SUM FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem presents some summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:If $(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1) \neq 0$ then*

(a)

$$\sum_{k=1}^{n} W_{-k}^3 = \frac{\Delta_4}{(rs - s^3 + 1)(r + s - 1)(r + s - rs + r^2 + s^2 + 1)}$$

where

$$\begin{aligned} \Delta_4 = {} & (2rs + s^3 - 1)W_{-n+1}^3 + (r^4 s + 3r^2 s^2 - r^3 s^3 + 2rs + r^3 + s^3 - 1)W_{-n}^3 - 3rs(r + s^2)W_{-n+1}^2 W_{-n} \\ & + 3rs^2 (rs - 1) W_{-n+1} W_{-n}^2 - (2rs + s^3 - 1)W_1^3 \\ & - (r^4 s + 3r^2 s^2 - r^3 s^3 + 2rs + r^3 + s^3 - 1)W_0^3 + 3rs(r + s^2)W_1^2 W_0 - 3rs^2 (rs - 1) W_0^2 W_1. \end{aligned}$$

(b)

$$\sum_{k=1}^{n} W_{-k+1}^2 W_{-k} = \frac{\Delta_5}{(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_5 &= -r(r+s^2)W_{-n+1}^3 - rs^3(r+s^2)W_{-n}^3 + (3r^2s^2+r^3+s^3-1)W_{-n+1}^2 W_{-n} \\ &\quad -s^2(2r^3-s^3+1)W_{-n}^2 W_{-n+1} + r(r+s^2)W_1^3 + rs^3(r+s^2)W_0^3 \\ &\quad -(3r^2s^2+r^3+s^3-1)W_1^2 W_0 + s^2(2r^3-s^3+1)W_0^2 W_1.\end{aligned}$$

(c)

$$\sum_{k=1}^{n} W_{-k}^2 W_{-k+1} = \frac{\Delta_6}{(rs-s^3+1)(r+s-1)(r+s-rs+r^2+s^2+1)}$$

where

$$\begin{aligned}\Delta_6 &= r(rs-1)W_{-n+1}^3 + rs^3(rs-1)W_{-n}^3 - s(2r^3-s^3+1)W_{-n+1}^2 W_{-n} \\ &\quad +(-2rs^4+r^4s+2rs+r^3+s^3-1)W_{-n}^2 W_{-n+1} - r(rs-1)W_1^3 \\ &\quad -rs^3(rs-1)W_0^3 + s(2r^3-s^3+1)W_1^2 W_0 - (-2rs^4+r^4s+2rs+r^3+s^3-1)W_0^2 W_1.\end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$s^3 W_{-n}^3 = W_{-n+2}^3 - 3rW_{-n+2}^2 W_{-n+1} + 3r^2 W_{-n+1}^2 W_{-n+2} - r^3 W_{-n+1}^3$$

and so

$$\begin{aligned}s^3W_{-n}^3 &= W_{-n+2}^3 - 3rW_{-n+2}^2W_{-n+1} + 3r^2W_{-n+1}^2W_{-n+2} - r^3W_{-n+1}^3 \\ s^3W_{-n+1}^3 &= W_{-n+3}^3 - 3rW_{-n+3}^2W_{-n+2} + 3r^2W_{-n+2}^2W_{-n+3} - r^3W_{-n+2}^3 \\ s^3W_{-n+2}^3 &= W_{-n+4}^3 - 3rW_{-n+4}^2W_{-n+3} + 3r^2W_{-n+3}^2W_{-n+4} - r^3W_{-n+3}^3 \\ &\vdots \\ s^3W_{-3}^3 &= W_{-1}^3 - 3rW_{-1}^2W_{-2} + 3r^2W_{-2}^2W_{-1} - r^3W_{-2}^3 \\ s^3W_{-2}^3 &= W_0^3 - 3rW_0^2W_{-1} + 3r^2W_{-1}^2W_0 - r^3W_{-1}^3 \\ s^3W_{-1}^3 &= W_1^3 - 3rW_1^2W_0 + 3r^2W_0^2W_1 - r^3W_0^3\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}s^3(\sum_{k=1}^{n} W_{-k}^3) &= (-W_{-n+1}^3 - W_{-n}^3 + W_1^3 + W_0^3 + \sum_{k=1}^{n} W_{-k}^3) \\ &\quad -3r(-W_{-n+1}^2W_{-n} + W_1^2W_0 + \sum_{k=1}^{n} W_{-k+1}^2W_{-k}) \\ &\quad +3r^2(-W_{-n}^2W_{-n+1} + W_0^2W_1 + \sum_{k=1}^{n} W_{-k}^2W_{-k+1}) \\ &\quad -r^3(-W_{-n}^3 + W_0^3 + \sum_{k=1}^{n} W_{-k}^3).\end{aligned} \tag{3.1}$$

Next we calculate $\sum_{k=1}^{n} W_{-k+1}^{2}W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$sW_{-n+1}^{2}W_{-n} = W_{-n+1}^{2}W_{-n+2} - rW_{-n+1}^{3}$$

and so

$$\begin{array}{rcl} sW_{-n+1}^{2}W_{-n} & = & W_{-n+1}^{2}W_{-n+2} - rW_{-n+1}^{3} \\ sW_{-n+2}^{2}W_{-n+1} & = & W_{-n+2}^{2}W_{-n+3} - rW_{-n+2}^{3} \\ sW_{-n+3}^{2}W_{-n+2} & = & W_{-n+3}^{2}W_{-n+4} - rW_{-n+3}^{3} \\ & \vdots & \\ sW_{-2}^{2}W_{-3} & = & W_{-2}^{2}W_{-1} - rW_{-2}^{3} \\ sW_{-1}^{2}W_{-2} & = & W_{-1}^{2}W_{0} - rW_{-1}^{3} \\ sW_{0}^{2}W_{-1} & = & W_{0}^{2}W_{1} - rW_{0}^{3} \end{array}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n} W_{-k+1}^{2}W_{-k} = (-W_{-n}^{2}W_{-n+1} + W_{0}^{2}W_{1} + \sum_{k=1}^{n} W_{-k}^{2}W_{-k+1}) - r(-W_{-n}^{3} + W_{0}^{3} + \sum_{k=1}^{n} W_{-k}^{3}). \quad (3.2)$$

Next we calculate $\sum_{k=1}^{n} W_{-k+1}^{2}W_{-k}$. Again using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$\begin{array}{rcl} s^{2}W_{-n}^{2} & = & W_{-n+2}^{2} - 2rW_{-n+2}W_{-n+1} + r^{2}W_{-n+1}^{2} \\ & \Rightarrow & s^{2}W_{-n}^{2}W_{-n+1} = W_{-n+2}^{2}W_{-n+1} - 2rW_{-n+1}^{2}W_{-n+2} + r^{2}W_{-n+1}^{3} \end{array}$$

and so

$$\begin{array}{rcl} s^{2}W_{-n}^{2}W_{-n+1} & = & W_{-n+2}^{2}W_{-n+1} - 2rW_{-n+1}^{2}W_{-n+2} + r^{2}W_{-n+1}^{3} \\ s^{2}W_{-n+1}^{2}W_{-n+2} & = & W_{-n+3}^{2}W_{-n+2} - 2rW_{-n+2}^{2}W_{-n+3} + r^{2}W_{-n+2}^{3} \\ s^{2}W_{-n+2}^{2}W_{-n+3} & = & W_{-n+4}^{2}W_{-n+3} - 2rW_{-n+3}^{2}W_{-n+4} + r^{2}W_{-n+3}^{3} \\ & \vdots & \\ s^{2}W_{-3}^{2}W_{-2} & = & W_{-1}^{2}W_{-2} - 2rW_{-2}^{2}W_{-1} + r^{2}W_{-2}^{3} \\ s^{2}W_{-2}^{2}W_{-1} & = & W_{0}^{2}W_{-1} - 2rW_{-1}^{2}W_{0} + r^{2}W_{-1}^{3} \\ s^{2}W_{-1}^{2}W_{0} & = & W_{1}^{2}W_{0} - 2rW_{0}^{2}W_{1} + r^{2}W_{0}^{3} \end{array}$$

If we add the above equations by side by, we get

$$s^2\sum_{k=1}^{n}W_{-k}^2W_{-k+1} = (-W_{-n+1}^2W_{-n}+W_1^2W_0+\sum_{k=1}^{n}W_{-k+1}^2W_{-k}) -2r(-W_{-n}^2W_{-n+1}+W_0^2W_1+\sum_{k=1}^{n}W_{-k}^2W_{-k+1}) +r^2(-W_{-n}^3+W_0^3+\sum_{k=1}^{n}W_{-k}^3). \quad (3.3)$$

Then, solving the system (3.1)-(3.2)-(3.3), the required results of (a),(b) and (c) follow.

Taking $r=s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.1. *If $r=s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}W_{-k}^3=\frac{1}{2}(W_{-n+1}^3+3W_{-n}^3-3W_{-n+1}^2W_{-n}-W_1^3-3W_0^3+3W_1^2W_0)$.

(b) $\sum_{k=1}^{n}W_{-k+1}^2W_{-k}=\frac{1}{2}(-W_{-n+1}^3-W_{-n}^3+2W_{-n+1}^2W_{-n}-W_{-n}^2W_{-n+1}+W_1^3+W_0^3-2W_1^2W_0+W_0^2W_1)$.

(c) $\sum_{k=1}^{n}W_{-k}^2W_{-k+1}=\frac{1}{2}(-W_{-n+1}^2W_{-n}+W_{-n}^2W_{-n+1}+W_1^2W_0-W_0^2W_1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0,F_1=1$).

Corollary 3.2. *For $n\geq 1$, Fibonacci numbers have the following properties.*

(a) $\sum_{k=1}^{n}F_{-k}^3=\frac{1}{2}(F_{-n+1}^3+3F_{-n}^3-3F_{-n+1}^2F_{-n}-1)$.

(b) $\sum_{k=1}^{n}F_{-k+1}^2F_{-k}=\frac{1}{2}(-F_{-n+1}^3-F_{-n}^3+2F_{-n+1}^2F_{-n}-F_{-n}^2F_{-n+1}+1)$.

(c) $\sum_{k=1}^{n}F_{-k}^2F_{-k+1}=\frac{1}{2}(-F_{-n+1}^2F_{-n}+F_{-n}^2F_{-n+1})$.

Taking $W_n=L_n$ with $L_0=2,L_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.3. *For $n\geq 1$, Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}L_{-k}^3=\frac{1}{2}(L_{-n+1}^3+3L_{-n}^3-3L_{-n+1}^2L_{-n}-19)$.

(b) $\sum_{k=1}^{n}L_{-k+1}^2L_{-k}=\frac{1}{2}(-L_{-n+1}^3-L_{-n}^3+2L_{-n+1}^2L_{-n}-L_{-n}^2L_{-n+1}+9)$.

(c) $\sum_{k=1}^{n}L_{-k}^2L_{-k+1}=\frac{1}{2}(-L_{-n+1}^2L_{-n}+L_{-n}^2L_{-n+1}-2)$.

Taking $r=2,s=1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.2. *If $r=2,s=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}W_{-k}^3=\frac{1}{14}(2W_{-n+1}^3+16W_{-n}^3-9W_{-n+1}^2W_{-n}+3W_{-n}^2W_{-n+1}-2W_1^3-16W_0^3+9W_1^2W_0-3W_0^2W_1)$.

(b) $\sum_{k=1}^{n}W_{-k+1}^2W_{-k}=\frac{1}{14}(-3W_{-n+1}^3-3W_{-n}^3+10W_{-n+1}^2W_{-n}-8W_{-n}^2W_{-n+1}+3W_1^3+3W_0^3-10W_1^2W_0+8W_0^2W_1)$.

(c) $\sum_{k=1}^{n}W_{-k}^2W_{-k+1}=\frac{1}{14}(W_{-n+1}^3+W_{-n}^3-8W_{-n+1}^2W_{-n}+12W_{-n}^2W_{-n+1}-W_1^3-W_0^3+8W_1^2W_0-12W_0^2W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0,P_1=1$).

Corollary 3.4. *For $n \geq 1,$ Pell numbers have the following properties.*

(a) $\sum_{k=1}^{n} P_{-k}^{3} = \frac{1}{14}(2P_{-n+1}^{3} + 16P_{-n}^{3} - 9P_{-n+1}^{2}P_{-n} + 3P_{-n}^{2}P_{-n+1} - 2).$

(b) $\sum_{k=1}^{n} P_{-k+1}^{2}P_{-k} = \frac{1}{14}(-3P_{-n+1}^{3} - 3P_{-n}^{3} + 10P_{-n+1}^{2}P_{-n} - 8P_{-n}^{2}P_{-n+1} + 3).$

(c) $\sum_{k=1}^{n} P_{-k}^{2}P_{-k+1} = \frac{1}{14}(P_{-n+1}^{3} + P_{-n}^{3} - 8P_{-n+1}^{2}P_{-n} + 12P_{-n}^{2}P_{-n+1} - 1).$

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.5. *For $n \geq 1,$ Pell-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n} Q_{-k}^{3} = \frac{1}{14}(2Q_{-n+1}^{3} + 16Q_{-n}^{3} - 9Q_{-n+1}^{2}Q_{-n} + 3Q_{-n}^{2}Q_{-n+1} - 96).$

(b) $\sum_{k=1}^{n} Q_{-k+1}^{2}Q_{-k} = \frac{1}{14}(-3Q_{-n+1}^{3} - 3Q_{-n}^{3} + 10Q_{-n+1}^{2}Q_{-n} - 8Q_{-n}^{2}Q_{-n+1} + 32).$

(c) $\sum_{k=1}^{n} Q_{-k}^{2}Q_{-k+1} = \frac{1}{14}(Q_{-n+1}^{3} + Q_{-n}^{3} - 8Q_{-n+1}^{2}Q_{-n} + 12Q_{-n}^{2}Q_{-n+1} - 48).$

Taking $r = 1, s = 2$ in Theorem 3.1 (a),(b) and (c), we obtain the following proposition.

Proposition 3.3. *If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}^{3} = \frac{1}{70}(-11W_{-n+1}^{3} - 18W_{-n}^{3} + 30W_{-n+1}^{2}W_{-n} - 12W_{-n}^{2}W_{-n+1} + 11W_{1}^{3} + 18W_{0}^{3} - 30W_{1}^{2}W_{0} + 12W_{0}^{2}W_{1}).$

(b) $\sum_{k=1}^{n} W_{-k+1}^{2}W_{-k} = \frac{1}{70}(5W_{-n+1}^{3} + 40W_{-n}^{3} - 20W_{-n+1}^{2}W_{-n} - 20W_{-n}^{2}W_{-n+1} - 5W_{1}^{3} - 40W_{0}^{3} + 20W_{1}^{2}W_{0} + 20W_{0}^{2}W_{1}).$

(c) $\sum_{k=1}^{n} W_{-k}^{2}W_{-k+1} = \frac{1}{70}(-W_{-n+1}^{3} - 8W_{-n}^{3} - 10W_{-n+1}^{2}W_{-n} + 18W_{-n}^{2}W_{-n+1} + W_{1}^{3} + 8W_{0}^{3} + 10W_{1}^{2}W_{0} - 18W_{0}^{2}W_{1}).$

From the last proposition, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.6. *For $n \geq 1,$ Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} J_{-k}^{3} = \frac{1}{70}(-11J_{-n+1}^{3} - 18J_{-n}^{3} + 30J_{-n+1}^{2}J_{-n} - 12J_{-n}^{2}J_{-n+1} + 11).$

(b) $\sum_{k=1}^{n} J_{-k+1}^{2}J_{-k} = \frac{1}{70}(5J_{-n+1}^{3} + 40J_{-n}^{3} - 20J_{-n+1}^{2}J_{-n} - 20J_{-n}^{2}J_{-n+1} - 5).$

(c) $\sum_{k=1}^{n} J_{-k}^{2}J_{-k+1} = \frac{1}{70}(-J_{-n+1}^{3} - 8J_{-n}^{3} - 10J_{-n+1}^{2}J_{-n} + 18J_{-n}^{2}J_{-n+1} + 1).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.7. *For $n \geq 1,$ Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} j_{-k}^{3} = \frac{1}{70}(-11j_{-n+1}^{3} - 18j_{-n}^{3} + 30j_{-n+1}^{2}j_{-n} - 12j_{-n}^{2}j_{-n+1} + 143).$

(b) $\sum_{k=1}^{n} j_{-k+1}^{2}j_{-k} = \frac{1}{70}(5j_{-n+1}^{3} + 40j_{-n}^{3} - 20j_{-n+1}^{2}j_{-n} - 20j_{-n}^{2}j_{-n+1} - 205).$

(c) $\sum_{k=1}^{n} j_{-k}^{2}j_{-k+1} = \frac{1}{70}(-j_{-n+1}^{3} - 8j_{-n}^{3} - 10j_{-n+1}^{2}j_{-n} + 18j_{-n}^{2}j_{-n+1} + 13).$

4 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can summarize the sections as follows:

- In section 1, we present some background about generalized Fibonacci numbers.
- In section 2, summation formulas have been presented for the generalized Fibonacci numbers with positive subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with positive subscripts have been given.
- In section 3, summation formulas have been presented for the generalized Fibonacci numbers with negative subscripts. As special cases, summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers with negative subscripts have been given.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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